Dynamic Macroeconomics

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Preliminary[1]

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Chapter 1

Models with no Uncertainty

This section introduces the main tool in quantitative macroeconomic research during the last decades: the neoclassical growth model. Even though the simplest version of this model is not good enough to explain most macroeconomic phenomena, it is a benchmark upon which most of the more complex models are built. Understanding the neoclassical growth model will give the tools to study questions in different fields, such as monetary policy, fiscal policy, unemployment, growth, and business cycles, among others.

The neoclassical growth model was a response to the Solow model, proposed in his 1957 paper, where he explained capital accumulation in the economy. In his model, Solow assumed that the economy was composed of agents that saved an exogenous fraction of their income, which was invested in capital. Even though this model has rich implications on growth and convergence between economies that start from different initial conditions, the assumption that agents save an exogenous and fixed fraction of their income is very strong. The first version of the neoclassical growth model was proposed in the decade of 1960s by David Cass and Tjalling Koopmans, where they assumed that, instead of saving a constant fraction of their income, households solve an intertemporal utility maximization problem, so savings and consumption are endogenous decisions that depend on prices. In particular, the savings decision depends on the interest rate which, in equilibrium, is determined by the aggregate level of capital in the economy, and by the discount factor, among other parameters.

It turns out that most of the models currently used in research are extensions of this model. Real business cycles and the short-term fluctuations in the economy are studied using

the neoclassical growth model with uncertainty, where there are shocks to productivity that drive fluctuations. Monetary economics use neokeynesian models, which are real business cycle models with frictions, such as sticky prices and wages. Models that study inequality and income distribution are composed of heterogeneous agents that interact in markets in much the same way as the representative agent does in the neoclassical growth model. Heterogeneity in these models comes in different forms: idiosyncratic risk faced by different agents, life-cycle models in which agents of different cohorts interact, or ex-ante differences such as initial wealth or ability. Similarly, models that study housing and financial markets are extensions of the neoclassical growth model, where individuals have access to multiple assets that differ in returns and risk.

But even thought the neoclassical growth model is such a useful tool, it turns out that by itself it is not good enough to study most of the questions. In order to successfully explain macroeconomic issues, certain extensions and assumptions must be made, depending on the issue of study.

The following sections will present the neoclassical growth model and some of its extensions, such as models with endogenous labor and exogenous technological growth. Every section will first study the Pareto-optimal allocations of the specific model by solving the social planner's problem. Then, every section will define and characterize a descentralized competitive equilibrium, where agents choose consumption, savings and labor by interacting in markets where prices are fixed competitively in equilibrium. In applications where there are no frictions, such as externalities, public goods, taxes, or private information, both the social planner's solution and the competitive equilibrium will coincide, as stated by the welfare theorems. However, in models with such frictions the solution to both problems will usually differ.

1.1 The Neoclassical Growth Model

In the economy there are two types of agents: households and firms. Households live for $T = \infty$ periods. Every period they choose consumption, investment and labor. Firms use labor and capital as inputs to produce the final good in the economy, which is devoted to consumption and investment.

We assume there is a large number of households and firms and all of them are identical.

We call this model a "representative agent" model because all households and firms take the same decision in equilibrium. We denote the population size as L_t. For simplicity, in this section we assume that there is a continuum of households of measure 1 and there is no population growth. This means that $L_t = 1$ for every period t.

The assumption for there being a large number of agents is that in real life there are millions of households and firms, and the effect of the decisions of a single agent have negligible impact on macroeconomic aggregates. For instance, according to Inegi, in Mexico there were 32 million households in 2015, so the actions of one household have negligible effects on the overall economy. We will keep the assumption of identical households for now. In later sections we will assume there are heterogeneous households.

For convention, we will denote aggregate variables with upper-case letters, and per-capita variables with lower-case letters. For example, C_t and c_t will denote aggregate and per-capita consumption, respectively, K_t and k_t are aggregate and per-capita capital, and so on. In this section, given that we assume population size equal to 1 every period, it turns out that $c_t = C_t, k_t = K_t, \ldots$, so we will express the model in per-capita terms. In later sections, where there is population growth, I will make the explicit the difference between aggregate and per-capita variables.

Households maximize a lifetime utility function that depends on consumption in every period of life $\{c_t\}_{t=0}^{\infty}$:

$$U(c_0, c_1, \ldots) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

where β is the discount factor and the period utility function u is strictly concave and satisfies the Inada conditions:

$$\begin{aligned} &1.\ u'(c) > 0 && 3.\ \lim_{c\to 0} u'(c) = \infty \\ &2.\ u''(c) < 0 && 4.\ \lim_{c\to\infty} u'(c) = 0 \end{aligned} \tag{1.1}$$

Every period, each agent within the household has one unit of time that is devoted inelastically to labor. In addition to time, households start their lives with an amount of capital k_0, which is devoted to production.

Total output in the economy y_t is produced by firms that hire labor l_t and capital k_t in

competitive markets as inputs to produce according to a neoclassical production function F, such that $y_t = F(k_t, l_t)$. We assume that F satisfies the following properties: 1) F is homogeneous of degree 1 -has constant returns to scale-, such that $F(\lambda k_t, \lambda l_t) = \lambda F(k_t, l_t)$ for every λ; 2) F is concave, such that $F_k, F_l > 0, F_{kk}, F_{ll} < 0$, and $F_{kl} > 0$; and 3) F satisfies the Inada conditions:

$$\begin{aligned} &1.\ \lim_{k\to 0} F_k(k,l) = \infty && 3.\ \lim_{k\to\infty} F_k(k,l) = 0 \\ &2.\ \lim_{l\to 0} F_l(k,l) = \infty && 4.\ \lim_{l\to\infty} F_l(k,l) = 0 \end{aligned} \tag{1.2}$$

Final good production y_t in the economy is devoted to consumption c_t and investment i_t, so the aggregate resource constraint is given by $y_t = c_t + i_t$.

Given an investment choice i_t, capital in the economy evolves according to:

$$k_{t+1} = (1-\delta)k_t + i_t$$

where δ is the depreciation rate of capital. This equation states that the amount of capital at $t+1$ is equal to the undepreciated capital at t plus the total investments made.

1.1.1 Social Planner's Problem

The social planner's problem, whose solution yields the set of Pareto-optimal allocations in this economy, is given by:

$$\max_{c_t, i_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t.$$

$$\forall t \in \{0, 1, \ldots\}: \quad c_t + i_t = F(k_t, 1) \tag{1.3}$$

$$k_{t+1} = (1-\delta)k_t + i_t \tag{1.4}$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given} \tag{1.5}$$

The social planner maximizes the utility of the representative agent in the economy subject to the resource constraint in the economy (1.3), the capital evolution equation (1.4), non-negativity constraints, and the initial level of capital in the economy k_0 given (1.5). Combining the first two constraints, yields a simplified version of the social planner's problem:

$$\max_{c_t, i_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{1.6}$$

$$\forall t \in \{0, 1, \ldots\}: \quad c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1)$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}$$

The constraint that the social planner faces is such that the total amount of the final good produced $F(k_t, 1)$ plus the undepreciated capital $(1-\delta)k_t$ should be either consumed c_t or allocated to capital for production next period k_{t+1}.

The social planner's problem (1.6) is a maximization problem with restrictions, so we can solve it by setting up the corresponding lagrangean:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + F(k_t, 1) - c_t - k_{t+1}\right)$$

Note that there is one resource constraint for every period t, so we assign a lagrange multiplier λ_t for every such restriction. The multiplier λ_t is called the **shadow price of capital**, and represents the marginal value of capital or, equivalently, the additional utility that loosening the resource constraint by one unit would generate to the household. In this lagreangean, we are ignoring the non-negativity constraints $c_t, k_{t+1} \geq 0$ because it will never be optimal for the social planner to assign consumption or capital equal to zero in equilibrium, given the Inada conditions on the utility and production functions, given by conditions (1.1) and (1.2). In equilibrium, if $c_t = 0$, given that the marginal utility of consumption is infinite, increasing consumption by a small amount will generate a large increase in utility so it is optimal for the social planner to assign $c_t > 0$. Similarly, in equilibrium $k_{t+1} > 0$, given that if $k_{t+1} = 0$ the marginal utility of increasing capital is so large that the planner will find it optimal to set $k_{t+1} > 0$.

To characterize the equilibrium, the first order conditions are:

$$[c_t]: \quad \beta^t u'(c_t) - \lambda_t = 0 \tag{1.7}$$

$$[k_{t+1}]: \quad -\lambda_t + \lambda_{t+1}\left(1 - \delta + F_k(k_{t+1}, 1)\right) = 0 \tag{1.8}$$

$$[\lambda_t]: \quad (1-\delta)k_t + F(k_t, 1) - c_t - k_{t+1} = 0 \tag{1.9}$$

By rearranging equations (1.7) and (1.8), we obtain the following optimality condition for the social planner's problem:

$$u'(c_t) = (1 - \delta + F_k(k_{t+1}, 1))\beta u'(c_{t+1}) \tag{1.10}$$

This condition is commonly known as the **Euler equation** or **intertemporal optimality condition**, as it illustrates the inter-temporal trade-off faced by households: allocating one more unit of consumption today c_t yields additional utility $u'(c_t)$, while allocating an additional unit of capital k_{t+1} yields additional utility in present value given by $(1 - \delta + F_k(k_{t+1}, 1))\beta u'(c_{t+1})$. The individual will choose every period t the amount of consumption c_t and capital k_{t+1} that yield exactly the same marginal utility. This equation can be rearranged in the following way:

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{MRS}} = \underbrace{(1 - \delta + F_k(k_{t+1}, 1))}_{\text{MRT}} \tag{1.11}$$

The left-hand side of equation (1.11) represents the marginal rate of subsitution of households, given the utility function u, and the right-hand side represents the marginal rate of transformation, given the production function F. The optimal choice equates the marginal rate of substitution and the marginal rate of transformation.

Given that the utility function is strictly concave and the constraint set is strictly convex, the first order conditions of the planner's problem are necessary conditions for an optimal solution. However, for these to be sufficient conditions we need to add the following **transversality condition** on k_{t+1}:

$$\lim_{t \to \infty} \lambda_t k_{t+1} = 0 \tag{1.12}$$

Stokey et al. (1989) show that the following are necessary and sufficient conditions for the allocations $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ to be optimal and solve the social planner's problem:

$$\textbf{Euler equation:} \qquad \frac{u'(c_t)}{\beta u'(c_{t+1})} = (1 - \delta + F_k(k_{t+1}, 1)), \quad \forall t \geq 0 \tag{1.13}$$

$$\textbf{Resource constraint:} \qquad c_t + k_{t+1} = (1 - \delta)k_t + F(k_t, 1), \quad \forall t \geq 0 \tag{1.14}$$

$$\textbf{Transversality condition:} \qquad \lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{1.15}$$

$$\textbf{Initial condition:} \qquad k_0 \text{ given} \tag{1.16}$$

Conditions (1.13) and (1.14) come directly from the first order conditions of the social planner's problem. Note that these equations for all t constitute a set of difference equations in k_t, k_{t+1} and k_{t+2}. For this system of difference equations to have a solution, we need two "boundary conditions". The transversality and initial conditions (1.15) and (1.16) play the role of these boundary conditions. A detailed derivation of the transversality condition and the proof for them to be necessary and sufficient conditions for an optimum are explained in Stokey et al. (1989).

Conditions (1.13)-(1.16) fully characterize the set of Pareto-optimal allocations in the economy. This means that if the allocations $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ are given to households in the economy, there is no way to improve a single household without affecting at least another one.

1.1.2 Competitive Equilibrium

This section characterizes a competitive equilibrium in this environment, where households and firms are left to interact in competitive markets and prices are fixed such that markets clear. The environment is the same as in last section, although here households have access to risk-free financial assets to save in addition to investments in capital.

Definition of Equilibrium

Every period, each agent within the household has one unit of time that is devoted inelastically to labor in exchange for a wage w_t per unit of time. In addition to labor income, households own capital in the economy k_t, which they rent to firms in exchange for a rental rate R_t per unit of capital. Every period the household can save in risk-free bonds a_t, with

net returns given by the real interest rate r_t. Finally, households own firms in the economy, so every period they receive the profits generated by the firm π_t. This means that every period the household faces the following budget constraint:

$$c_t + i_t + a_{t+1} = w_t + R_t k_t + (1 + r_t) a_t + \pi_t$$

where i_t denotes investment, w_t is the wage per unit of labor, R_t is the rental price of capital, r_t is the real interest rate on risk-free bonds π_t are the profits of the firm. Note that in this budget constraint the price of the final good is normalized to one or, equivalently, the budget constraint is expressed in terms of the final good. In particular, wages w_t and capital rents r_t are in terms of the final good.

As in last section, investment is denoted by i_t, such that capital evolves according to:

$$k_{t+1} = (1 - \delta) k_t + i_t$$

where δ is the depreciation rate of capital. The amount of capital at $t + 1$ is equal to the undepreciated capital at t plus the total investments made by the household.

The household's problem is given by:

$$\max_{\substack{c_t, k_{t+1}, i_t \\ t \geq 0}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{1.17}$$

$$c_t + i_t + a_{t+1} = w_t + R_t k_t + (1 + r_t) a_t + \pi_t \tag{1.18}$$

$$k_{t+1} = (1 - \delta) k_t + i_t \tag{1.19}$$

$$\lim_{T \to \infty} \left(\prod_{t=0}^{T} \frac{1}{1 + r_t} \right) a_{T+1} \geq 0 \tag{1.20}$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given} \tag{1.21}$$

Note that there is one budget constraint (1.18) and capital evolution constraint (1.19) for every period t. The constraint given by equation (1.20) is called a **No-Ponzi condition**, and ensures that individuals do not get accumulate increasingly large levels of debt in the long run. Without this condition, individuals could roll-over their debt period after period by borrowing a larger amount every time. This condition rules out those cases.

Given the production function F, the firm chooses the amount of capital and labor to

hire every period to maximize profits:

$$\begin{aligned} \pi_t &= \max_{k_t, l_t} \quad y_t - w_t l_t - R_t k_t \\ &= \max_{k_t, l_t} \quad F(k_t, l_t) - w_t l_t - R_t k_t \end{aligned} \tag{1.22}$$

Definition 1 (Competitive Equilibrium)**.** *A competitive equilibrium are allocations for the household* $\{c_t, k^s_{t+1}, i_t, a_{t+1}\}_{t=0}^{\infty}$*, allocations for the firm* $\{k^d_t, l_t\}_{t=0}^{\infty}$*, and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ *such that:*

1. *Given* k_0 *and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$*, the allocations* $\{c_t, k^s_{t+1}, i_t, a_{t+1}\}_{t=0}^{\infty}$ *solve the optimization problem of the household, described by (1.17).*

2. *At every* $t \in \{0, 1, \ldots\}$*, given prices* w_t, R_t, r_t*, the allocations* k^d_t, l_t *solve the optimization problem of the firm, described by (1.22).*

3. *Markets clear for all* $t \geq 0$*:*

   (a) *Goods:* $c_t + i_t = F(k^d_t, l_t)$

   (b) *Labor:* $l_t = 1$

   (c) *Capital:* $k^d_t = k^s_t$

   (d) *Risk-free bonds:* $a_t = 0$

The market clearing condition for labor states that, in equilibrium, total demand for labor by the firms should be equal to one. This is because we assumed that total population size is equal to one and households supply labor inelastically. Later sections will assume that there is endogenous labor supply, which will change this market clearing condition. The last market clearing condition states that, in equilibrium, there is **zero net supply** for risk-free bonds. This is because we have assumed that in the economy there is a representative agent. Given that in equilibrium the total amount saved must equal total amount borrowed, and all agents are equal to each other, this can only happen if the representative household has no savings. If there were more than one agent in the economy (as will be the case in Section **??**), it could be the case that, in equilibrium, $a^i_{t+1} > 0$ and $a^j_{t+1} < 0$ for agents $i \neq j$, so that agent i is a saver, while agent j is a borrower.

To solve for an equilibrium, it suffices to solve problems (1.17) and (1.22) and set prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ that clear the markets in the economy. Next subsections solve the household's and firm's problem.

Household's Problem

Let's start by solving the problem of the household. First of all, notice that, instead of solving problem (1.17), we can combine the first two restrictions such that the budget constraint becomes:

$$c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t$$

This means that the household's problem can be expressed as:

$$\max_{c_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{1.23}$$

$$\begin{aligned}
\forall t \in \{0,1,\ldots\}: \quad & c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t \\
& c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given} \\
& \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0
\end{aligned}$$

In this problem, we got rid of investment i_t by combining the restrictions, so the households only chooses consumption c_t, capital k_{t+1} and risk-free bonds a_{t+1}. Once we solve for the equilibrium, the sequence for investment can be recovered using the capital evolution equation:

$$i_t = k_{t+1} - (1-\delta)k_t$$

As with the social planner's problem, given the Inada conditions for the utility and production functions, in equilibrium the individual will never choose to optimally consume $c_t = 0$ or set capital $k_{t+1} = 0$. Given that the marginal utility of consumption is infinite when $c_t = 0$, it is always optimal to consume a positive amount. On the other hand, if the household sets $k_{t+1} = 0$, the Inada condition on F implies that the real interest rate $r_{t+1} = \infty$. Therefore, it is optimal for the household to choose a positive amount of capital k_{t+1}. This means that we can safely ignore the non-negativity constraints when solving the

problem. Similarly, we can ignore the No-Ponzi condition for now.

The associated lagrangean is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t - c_t - k_{t+1} - a_{t+1}\right)$$

There is a lagrange multiplier λ_t for the budget constraint at each period, which represents the shadow price of one unit of wealth, and is interpreted as the additional utility that would bring to the individual having an additional unit of income in equilibrium. The first order conditions for this problem are:

$$[c_t]: \quad \beta^t u'(c_t) - \lambda_t = 0 \tag{1.24}$$

$$[k_{t+1}]: \quad -\lambda_t + \lambda_{t+1}\left(1 - \delta + R_{t+1}\right) = 0 \tag{1.25}$$

$$[a_{t+1}]: \quad -\lambda_t + \lambda_{t+1}\left(1 + r_{t+1}\right) = 0 \tag{1.26}$$

$$[\lambda_t]: \quad (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t + \pi_t - c_t - k_{t+1} - a_{t+1} = 0 \tag{1.27}$$

To obtain equations (1.25) and (1.26), note that k_{t+1} and a_{t+1} appear in two consecutive budget constraints: the one associated with period t and the one for period $t+1$. Equation (1.27) is simply the period t budget constraint. Combining the first three conditions yields:

$$\beta^t u'(c_t) = (1 - \delta + R_{t+1})\beta^{t+1} u'(c_{t+1}) \tag{1.28}$$

$$\beta^t u'(c_t) = (1 + r_{t+1})\beta^{t+1} u'(c_{t+1}) \tag{1.29}$$

Equations (1.28) and (1.29) imply that, in equilibrium, $R_t - \delta = r_t$, which is a **non-arbitrage condition**: the returns of capital must equal returns of risk-free bonds in equilibrium. In equilibrium, the household is indifferent between investing in capital or in risk-free bonds. This happens because if, for example, the returns to bonds were larger than returns to capital, households would choose to invest only in bonds and nothing in capital. The Inada condition for capital implies that $R_t = \infty$, so households have incentives to increase capital until both returns to capital and bonds are equal to each other.

Rearranging (1.28):

$$\underbrace{\frac{u'(c_t)}{\beta u'(c_{t+1})}}_{\text{MRS}} = \underbrace{(1-\delta+R_{t+1})}_{\text{MRT}} \tag{1.30}$$

This equation is commonly known as the **Euler equation** or the **intertemporal optimality condition** of the household, and has the same interpretation as the Euler condition for the social planner.

Given that the utility function is strictly concave and the constraint set is strictly convex, the first order conditions for the household's problem are necessary conditions for an optimal solution. However, for these to be sufficient conditions we need to add the following two transversality conditions on k_t and a_t:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0 \tag{1.31}$$

$$\lim_{t\to\infty} \lambda_t a_{t+1} = 0 \tag{1.32}$$

As for the social planner's problem, it can be shown that equations (1.27), (1.28), (1.29), (1.31) and (1.32) are necessary and sufficient conditions for an optimum, so any allocation that satisfies them constitutes a solution to the household's problem.

This means that, given a sequence of prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, the allocations $\{c_t, k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$ that satisfy the following equations characterize the optimal solution for the household:

$$\textbf{Euler eq.:} \quad \beta^t u'(c_t) = (1 - \delta + R_{t+1})\beta^{t+1} u'(c_{t+1}), \quad \forall t \geq 0 \tag{1.33}$$

$$\beta^t u'(c_t) = (1 + r_{t+1})\beta^{t+1} u'(c_{t+1}), \quad \forall t \geq 0 \tag{1.34}$$

$$\textbf{Bud. const.:} \quad c_t + k_{t+1} + a_{t+1} = (1 - \delta)k_t + w_t + R_t k_t + (1 + r_t)a_t + \pi_t, \quad \forall t \geq 0 \tag{1.35}$$

$$\textbf{Transv. cond.:} \quad \lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{1.36}$$

$$\lim_{t \to \infty} \beta^t u'(c_t) a_{t+1} = 0 \tag{1.37}$$

$$\textbf{Init. cond.:} \quad k_0, a_0 \text{ given} \tag{1.38}$$

Firm's Problem

Now, let's solve the firm's problem. Every period t, the problem of the firm is:

$$\max_{k_t, l_t} F(k_t, l_t) - w_t l_t - R_t k_t$$

Note that the firm's problem is completely static. This is because we are assuming that the firm does not own the capital, so it does not have to make any capital accumulation decisions. The dynamic, or inter-temporal, decision is completely made by the household. Extensions of this model that focus on studying corporate finance issues assume that firms have inter-temporal decisions, such as long term debt, equity issuance, and investment decisions.

Given the problem of the firm in the current version of the model, the first order conditions yield:

$$w_t = F_l(k_t, l_t) \tag{1.39}$$

$$R_t = F_k(k_t, l_t) \tag{1.40}$$

Given these conditions, in equilibrium firms make zero profits, so $\pi_t = 0$. This is a conse-

quence of the constant returns to scale assumption, which states that, for every λ:

$$F(\lambda k_t, \lambda l_t) = \lambda F(k_t, l_t)$$

Differentiate the above equation with respect to λ:

$$k_t F_k(\lambda k_t, \lambda l_t) + l_t F_l(\lambda k_t, \lambda l_t) = F(k_t, l_t)$$

This holds, in particular, for $\lambda = 1$, which means that:

$$F(k_t, l_t) = k_t \underbrace{F_k(k_t, l_t)}_{R_t} + l_t \underbrace{F_l(k_t, l_t)}_{w_t} = w_t l_t + R_t k_t \tag{1.41}$$

So $\pi_t = F(k_t, l_t) - w_t l_t - R_t k_t = 0$. In this way, the equilibrium is fully characterized by prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, and allocations $\{c_t, k_{t+1}^s, i_t, a_{t+1}\}_{t=0}^{\infty}$ for the household and $\{k_t^d, l_t\}_{t=0}^{\infty}$ for the firm, that satisfy equations (1.33)-(1.38) of the household, (1.39), (1.40), and (1.41) of the firm, and the market clearing conditions in Definition 4.

Welfare Theorems

Note the similarities between the optimality conditions of the social planner's problem, and the ones associated to the household's problem. It is easy to prove that the allocations of both problems coincide, by noting that, in the competitive equilibrium, the solution to the firm's problem yields $R_t = F_k(k_t, 1)$, given by equation (1.40), and the fact that $l_t = 1$, according to the labor market clearing condition. Similarly, using the solution to the firm's problem and the fact that $\pi_t = 0$, as shown by equation (1.41), the equivalence between the household's budget constraint (1.35) and the resource constraint (1.14) can be shown.

It is not surprising that the solutions to the social planner's problem and the competitive equilibrium are the same. This is because in this environment there are no frictions, such as externalities, public goods, taxes, or information problems, so the two welfare theorems hold. For completeness, I state the two welfare theorems:

Theorem 1 (First Welfare Theorem)**.** *Let $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t\}_{t=0}^{\infty}$ be a competitive equilibrium allocation. Then $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t\}_{t=0}^{\infty}$ is Pareto efficient.*

This theorem states that, whenever we solve for a competitive equilibrium, the resulting allocations are Pareto efficient, which means that those allocations also solve the social

planner's problem. In this way, whenever we want to find Pareto-optimal allocations, we can solve either the social planner's problem or the competitive equilibrium.

Theorem 2 (Second Welfare Theorem)**.** *Let* $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t\}_{t=0}^{\infty}$ *be a Pareto-optimal allocation. Then, there exist prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ *such that* $\{c_t, k_{t+1}, i_t, a_{t+1}, l_t, w_t, R_t, r_t\}_{t=0}^{\infty}$ *constitute a competitive equilibrium.*

This means that, whenever we want to solve for an equilibrium, we can explicitly solve the household's and firm's problems as in Definition 4, or we can solve the social planner's problem and construct a sequence of prices accordingly.

1.1.3 Steady state

A steady state in this economy is a competitive equilibrium (Definition 4) in which all variables per capita remain constant across time. Given that population size is constant and is equal to one, aggregate per capita variables are equal, in equilibrium, to the individual choices of households. In a steady state:

$$\begin{aligned} c_t &= c_{t+1} = c_{t+2} = \ldots = c^* \\ k_t &= k_{t+1} = k_{t+2} = \ldots = k^* \\ y_t &= y_{t+1} = y_{t+2} = \ldots = y^* \\ &\vdots \end{aligned}$$

Having solved for the equilibrium, we now can characterize how a steady state would look like. Take the Euler equation in the social planner's problem (1.10), and set all variables to their steady state values:

$$\begin{aligned} \frac{u'(c^*)}{\beta u'(c^*)} = (1 - \delta + F_k(k^*, 1)) \quad &\iff \quad \frac{1}{\beta} = (1 - \delta + F_k(k^*, 1)) \\ &\iff \quad F_k(k^*, 1) = \frac{1}{\beta} - 1 + \delta \end{aligned} \tag{1.42}$$

Note that in this equation everything is constant except for capital, so capital in steady state k^* is pinned down by (1.42). Moreover, if the production function F is strictly concave, it can be shown that this steady state is unique. This means that there is a unique k^*

which solves equation (1.42). Once the capital in steady state k^* is characterized, it is straightforward to find steady state values of all the other variables in the model. For production and consumption we can use the production function and resource constraint:

$$y^* = F(k^*, 1) \tag{1.43}$$

$$c^* = (1-\delta)k^* + F(k^*, 1) - k^* = F(k^*, 1) - \delta k^* \tag{1.44}$$

Similarly, prices can be computed using the solution to the firm's problem:

$$w^* = F_l(k^*, 1) \tag{1.45}$$

$$R^* = F_k(k^*, 1) \tag{1.46}$$

$$r^* = R^* - \delta \tag{1.47}$$

Note that in this environment there is no growth on per capita variables in steady state. This means that this model cannot successfully explain long-term growth of countries. Further sections will expand the model such that the economy can experience growth in the long run.

1.1.4 The Log Utility Case

There are few cases in which these models can be solved analytically with a closed-form solution. This section presents one such particular case. In most other cases, it is not possible to derive a closed form solution for equilibria and the only option is to solve the model numerically. Section 1.5 is an introduction to solving these models with a computer.

In this example, let the utility function be given by $u(c) = \log(c)$ and the aggregate production function be $F(k, l) = k^\alpha l^{1-\alpha}$. Assume that there is full depreciation, so $\delta = 1$.

Using equation (1.33), the intertemporal optimality condition is given by:

$$\frac{c_{t+1}}{\beta c_t} = R_{t+1}$$

This implies that consumption over time behaves according to:

$$c_{t+1} = R_{t+1}\beta c_t \tag{1.48}$$

Recursively plugging in equation (1.48) we obtain an expression for period t consumption:

$$c_1 = R_1 \beta c_0$$

$$c_2 = R_2 \beta c_1 = R_1 \cdot R_2 \cdot \beta^2 c_0$$

$$\dots$$

$$c_t = \left(\beta^t \Pi_{j=1}^t R_j\right) c_0 \tag{1.49}$$

If we take the budget constraints in equation (1.35) for periods $t, t+1, t+2 \dots$ and recursively combine them:

$$c_t + k_{t+1} = w_t + R_t k_t$$

$$c_{t+1} + k_{t+2} = w_{t+1} + R_{t+1} k_{t+1}$$

$$c_{t+2} + k_{t+3} = w_{t+2} + R_{t+2} k_{t+2}$$

$$\dots$$

We obtain:

$$k_{t+1} = \frac{1}{R_{t+1}} \left(c_{t+1} + k_{t+2} - w_{t+1}\right)$$

$$k_{t+2} = \frac{1}{R_{t+2}} \left(c_{t+2} + k_{t+3} - w_{t+2}\right)$$

$$\dots$$

Then:

$$
\begin{aligned}
& c_t + \frac{1}{R_{t+1}}\left(c_{t+1} + k_{t+2} - w_{t+1}\right) = w_t + r_t k_t \\
\iff \quad & c_t + \frac{c_{t+1}}{R_{t+1}} + \frac{k_{t+2}}{R_{t+1}} = w_t + \frac{w_{t+1}}{R_{t+1}} + r_t k_t \\
\iff \quad & c_t + \frac{c_{t+1}}{R_{t+1}} + \frac{(c_{t+2} + k_{t+3} - w_{t+2})}{R_{t+1} \cdot R_{t+2}} = w_t + \frac{w_{t+1}}{R_{t+1}} + r_t k_t \\
\iff \quad & c_t + \frac{c_{t+1}}{R_{t+1}} + \frac{c_{t+2}}{R_{t+1} \cdot R_{t+2}} + \frac{k_{t+3}}{R_{t+1} \cdot R_{t+2}} = w_t + \frac{w_{t+1}}{R_{t+1}} + \frac{w_{t+2}}{R_{t+1} \cdot R_{t+2}} + r_t k_t \\
\iff \quad & \ldots \text{ (Summing up to period } t+s) \\
\iff \quad & \sum_{l=0}^{s} \frac{c_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \frac{k_{t+s+1}}{\Pi_{j=t}^{t+s} R_j} = \sum_{l=0}^{s} \frac{w_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \quad k_t \\
\iff \quad & \ldots \text{ (Taking limit as } s \to \infty) \\
\iff \quad & \sum_{l=0}^{\infty} \frac{c_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \lim_{s\to\infty} \left(\frac{1}{\Pi_{j=t}^{t+s} R_j}\right) k_{t+s+1} = \sum_{l=0}^{\infty} \frac{w_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + \quad k_t \\
\iff \quad & \ldots \text{ (By transversality condition)} \\
\iff \quad & \underbrace{\sum_{l=0}^{\infty} \frac{c_{t+l}}{\Pi_{j=0}^{l} R_{t+j}}}_{\text{Lifetime consumption}} = \underbrace{\sum_{l=0}^{\infty} \frac{w_{t+l}}{\Pi_{j=0}^{l} R_{t+j}}}_{\text{Lifetime labor income}} + \quad k_t
\end{aligned}
$$

The transversality condition is used because:

$$
\begin{aligned}
\lim_{s\to\infty} \lambda_{t+s} k_{t+s+1} &= \lim_{s\to\infty} \beta^{t+s} u'(c_{t+s}) k_{t+s+1} \\
&= \lim_{s\to\infty} \frac{\beta^{t+s}}{c_{t+s}} k_{t+s+1} \\
&= \lim_{s\to\infty} \frac{\beta^{t+s}}{\beta^s \Pi_{j=t}^{t+s} R_j c_t} k_{t+s+1} \\
&= \left(\frac{\beta^t}{c_t}\right) \cdot \lim_{s\to\infty} \left(\frac{1}{\Pi_{j=t}^{t+s} R_j}\right) k_{t+s+1} = 0
\end{aligned}
$$

Plugging in the expression for consumption at period t given by (1.49):

$$\begin{aligned}
\sum_{l=0}^{\infty} \frac{c_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} &= \sum_{l=0}^{\infty} \frac{\beta^l \Pi_{j=1}^{l} R_{t+j} c_t}{\Pi_{j=0}^{l} R_{t+j}} \\
&= \frac{(1-\beta)}{R_t} c_t = \sum_{l=0}^{\infty} \frac{w_{t+l}}{\Pi_{j=0}^{l} R_{t+j}} + k_t \\
\iff \quad c_t &= \frac{1}{(1-\beta)} \left(\sum_{l=0}^{\infty} \frac{w_{t+l}}{\Pi_{j=1}^{l} R_{t+j}} + R_t k_t \right)
\end{aligned}$$

In particular:

$$c_0 = \frac{1}{(1-\beta)} \left(\sum_{t=0}^{\infty} \frac{w_t}{\Pi_{j=1}^{t} R_j} + R_0 k_0 \right) \tag{1.50}$$

$$c_t = \frac{1}{(1-\beta)} \left(\sum_{t=0}^{\infty} \frac{w_t}{\Pi_{j=1}^{t} R_j} + R_0 k_0 \right) \tag{1.51}$$

This is Milton Friedman's (1957) Permanent Income hypothesis. In this model, agents consume every period a fraction of total lifetime income. Instead of making the consumption decision based on present income, they choose how much to consume based on their lifetime income. This means that, in this model, additional income in any single period of life increases consumption at every other period. Note that equation (1.51) illustrates the path of consumption over life. Recall that the discount factor of households is given by a parameter ρ, where:

$$\beta = \frac{1}{1+\rho}$$

Also, $\beta = 1/R_j$ if and only if $\rho = R_j - 1 = r_j$. This means that the path of consumption of households is constant if and only if the discount rate of households equals the discount rate of the markets. If $\beta > 1/R_j = 1/(1+r_j)$, the household is more patient than the market, so consumption follows an increasing path over life. If, instead, $\beta < 1/R_j = 1/(1+r_j)$, households are more impatient than the market, so consumption will decrease over time.

Using the Euler equation (1.48) and the fact that $R_t = F_k(k_t, 1)$, we can see that for a

steady state to exist, the following condition must hold:

$$c^* = F_k(k^*, 1)\beta c^* \quad \Longleftrightarrow \quad 1 = F_k(k^*, 1) \cdot \beta$$

With the Cobb-Douglas functional form:

$$F_k(k, l) = \alpha \left(\frac{k}{l}\right)^{\alpha - 1}$$

So the steady state level of capital is given by:

$$\alpha k^{*\alpha - 1} = \frac{1}{\beta} \quad \Longleftrightarrow \quad k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

The other steady state variables are:

$$\begin{aligned} y_t = F(k_t, 1) = k^\alpha \quad \Longrightarrow \quad y^* &= (\alpha\beta)^{\frac{\alpha}{1-\alpha}} \\ R^* &= \alpha \left(\frac{1}{\alpha\beta}\right) \\ w^* &= (1-\alpha)\,(\alpha\beta)^{\frac{\alpha}{1-\alpha}} \\ c^* &= w^* + r^* + k^* - k^* = w^* + R^* \end{aligned}$$

1.2 Population Growth

So far, the model presented does not have long-run growth. When the economy reaches its steady state, all aggregate variables stay constant and do not grow. However, this is not what we observe in reality. Countries that are close to their steady state, such as most developed economies, experience growth every year. This section is a first step towards modelling an economy with long-run growth.

The model setup is exactly the same as before, with the only difference that now every household has L_t members, and family size grows at a rate n, such that:

$$L_{t+1} = (1+n)L_t$$

As in last section, we will denote per-capita variables in lower case, and absolute variables in upper case, such that:

$$c_t = \frac{C_t}{L_t}, \quad i_t = \frac{I_t}{L_t}, \quad k_t = \frac{K_t}{L_t}, \quad y_t = \frac{Y_t}{L_t}, \quad \ldots$$

Now, households maximize a lifetime utility function that depends on per-capita consumption rather than on aggregate consumption:

$$\sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}\right) = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right)$$

1.2.1 Social Planner's Problem

The social planner's problem is given by:

$$\begin{aligned}
\max_{C_t, K_{t+1}} \quad & \sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}\right) \quad s.t. \\
\forall t \in \{0,1,\ldots\}: \quad & C_t + K_{t+1} = (1-\delta)K_t + F(K_t, L_t) \\
& C_t, K_{t+1} \geq 0, \quad K_0 \text{ given}
\end{aligned}$$

By the constant-returns-to-scale property of the production function F, note that:

$$F(K_t, L_t) = L_t \cdot F\left(\frac{K_t}{L_t}, 1\right) = L_t F(k_t, 1)$$

The resource constraint can be re-expressed in per-capita terms by dividing both sides by L_t:

$$\begin{aligned}
& \frac{C_t}{L_t} + \frac{K_{t+1}}{L_t} = (1-\delta)\frac{K_t}{L_t} + F\left(\frac{K_t}{L_t}, 1\right) \\
\Leftrightarrow \quad & \frac{C_t}{L_t} + \frac{L_{t+1}}{L_t} \cdot \frac{K_{t+1}}{L_{t+1}} = (1-\delta)\frac{K_t}{L_t} + F\left(\frac{K_t}{L_t}, 1\right) \\
\Leftrightarrow \quad & c_t + (1+n)k_{t+1} = (1-\delta)k_t + F\left(k_t, 1\right)
\end{aligned}$$

The social planner's problem can be rewritten in per-capita terms as:

$$\begin{aligned}
\max_{c_t, k_{t+1}} \quad & \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) \quad s.t. \\
\forall t \in \{0, 1, \ldots\}: \quad & c_t + (1+n)k_{t+1} = (1-\delta)k_t + F\left(k_t, 1\right) \\
& c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}
\end{aligned} \tag{1.52}$$

The constraint that the social planner faces is such that per-capita output produced $F(k_t, 1)$ plus the undepreciated capital $(1-\delta)k_t$ is either consumed c_t or allocated to capital for production next period $(1+n)k_{t+1}$. The term $(1+n)$ takes into account that the population will grow from t to $t+1$ at a rate n, so if the economy wants to have k_{t+1} per-capita units of capital per capita in $t+1$, it has to invest $(1+n)k_{t+1}$ at t.

The lagrangean associated to problem (1.52) is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u\left(c_t\right) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + F(k_t, 1) - c_t - (1+n)k_{t+1}\right)$$

Again, we are ignoring the non-negativity constraints because the Inada conditions ensure that the social planner will choose positive consumption and capital per capita in equilib-

rium. The first order conditions are:

$$
\begin{aligned}
[c_t]: &\quad \beta^t u'(c_t) - \lambda_t = 0 \\
[k_{t+1}]: &\quad -\lambda_t(1+n) + \lambda_{t+1}\left(1 - \delta + F_k(k_t, 1)\right) = 0 \\
[\lambda_t]: &\quad (1-\delta)k_t + F(k_t, 1) - c_t - (1+n)k_{t+1} = 0
\end{aligned}
$$

By rearranging the conditions associated to c_t and k_{t+1}, we obtain the following optimality condition:

$$
\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta + F_k(k_{t+1}, 1))}{(1+n)} \tag{1.53}
$$

This is the Euler condition for the social planner's problem. The right-hand side differs from the Euler equation of the social planner without population growth because now the marginal product of capital k_{t+1} chosen at t has to be split between a larger population at $t+1$. The solution to the social planner's problem is characterized by allocations $\{c_t, k_{t+1}\}_{t=0}^{\infty}$ that satisfy the following necessary and sufficient conditions for an optimum:

$$
\textbf{Euler eq.:} \qquad \frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta + F_k(k_{t+1}, 1))}{(1+n)}, \quad \forall t \geq 0 \tag{1.54}
$$

$$
\textbf{Resource constraint:} \qquad c_t + (1+n)k_{t+1} = (1-\delta)k_t + F(k_t, 1), \quad \forall t \geq 0 \tag{1.55}
$$

$$
\textbf{Transv. cond.:} \qquad \lim_{t \to \infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{1.56}
$$

$$
\textbf{Initial condition:} \qquad k_0 \text{ given} \tag{1.57}
$$

Note the similarity between conditions (1.54)-(1.57) and (1.13)-(1.16). In both cases, variables are expressed in per-capita terms. However, recall that in Section 1.1 there was no population growth and population size was normalized to 1, so the optimality conditions did not depend on n. In this section, there is population growth, so the Euler equation depends on the growth rate n. However, if we set population growth $n = 0$, we are back to the original optimality conditions.

1.2.2 Competitive Equilibrium

As in Section 1.1, we assume there is inelastic labor supply so every period each agent within the household supplies one unit of time to labor. The only difference is that now households have L_t agents, all of which supply labor, so total labor income equals $w_t L_t$. Households can invest in capital K_t, which has returns equal to R_t, and in risk-free bonds A_t that have net returns equal to the interest rate r_t. The households are the owners of the firms so every period they receive profits equal to Π_t. However, as shown in equation (1.41), profits are equal to zero in equilibrium whenever the production function satisfies the constant-returns-to-scale assumption and the firm operates in perfect competition. This will be the case throughout most of these lecture notes, so we will omit henceforth the profits Π_t from the households' problem. Every period, the household faces the following budget constraint:

$$C_t + I_t + A_{t+1} = w_t L_t + R_t K_t + (1 + r_t) A_t$$

Combining the budget constraint with the capital evolution equation, $K_{t+1} = (1-\delta)K_t + I_t$, and expressing the variables in per-capita terms:

$$\begin{aligned}
& \frac{C_t}{L_t} + \frac{K_{t+1}}{L_t} + \frac{A_{t+1}}{L_t} = (1-\delta)\frac{K_t}{L_t} + w_t + R_t \frac{K_t}{L_t} + (1+r_t)\frac{A_t}{L_t} \\
\iff \quad & \frac{C_t}{L_t} + \frac{L_{t+1}}{L_t} \cdot \frac{K_{t+1}}{L_{t+1}} + \frac{L_{t+1}}{L_t} \cdot \frac{A_{t+1}}{L_{t+1}} = (1-\delta)\frac{K_t}{L_t} + w_t + R_t \frac{K_t}{L_t} + (1+r_t)\frac{A_t}{L_t} \\
\iff \quad & c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t
\end{aligned}$$

The problem of the household can be conveniently expressed in per-capita terms:

$$\max_{c_t, k_{t+1}, a_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{1.58}$$

$$\begin{aligned}
\forall t \in \{0, 1, \ldots\} : \quad & c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given} \\
& \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} \geq 0
\end{aligned}$$

Given the production function, the firm chooses the amount of capital K_t and labor L_t to hire every period to maximize profits:

$$\Pi_t = \max_{K_t, L_t} F(K_t, L_t) - w_t L_t - R_t k_t \tag{1.59}$$

Note that the household maximizes consumption per-capita, so it is more convenient to express its problem in per-capita terms. In contrast, the firm's problem cannot be stated in per-capita terms, because the firm chooses absolute levels of capital and labor, and not only the ratio of capital per-worker. As will be seen later in this section, to compute the steady state, we will re-write the optimality conditions of the firm in per-capita terms.

The following is the definition of a competitive equilibrium in this environment, where there is population growth.

Definition 2 (Competitive Equilibrium)**.** *A competitive equilibrium are allocations for the household* $\{c_t, k^s_{t+1}, i_t, a_{t+1}\}_{t=0}^{\infty}$*, allocations for the firm* $\{K^d_t, L^d_t\}_{t=0}^{\infty}$*, and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ *such that:*

1. *Given* k_0*,* a_0*, and prices* $\{w_t, R_t, r_t\}_{t=0}^{\infty}$*, the allocations* $\{c_t, k^s_{t+1}, i_t, a_{t+1}\}_{t=0}^{\infty}$ *solve the optimization problem of the household described by (1.58).*

2. *At every* $t \in \{0, 1, \ldots\}$*, given prices* w_t, R_t, r_t*, the allocations* K^d_t, L^d_t *firms solve the optimization problem of the firm, described by (1.59)*

3. *Markets clear for all* $t \geq 0$*:*

   (a) *Goods:* $c_t + i_t = y_t$

   (b) *Labor:* $L^d_t = L_t$

   (c) *Capital:* $K^d_t = K^s_t$

   (d) *Risk-free bonds:* $a_t = 0$

Solving for the equilibrium

The lagrangean associated to the household's problem (1.58) is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u(c_t) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1}\right)$$

The first order conditions for this problem are:

$$[c_t]: \quad \beta^t u'(c_t) - \lambda_t = 0 \tag{1.60}$$

$$[k_{t+1}]: \quad -\lambda_t(1+n) + \lambda_{t+1}(1-\delta+R_{t+1}) = 0 \tag{1.61}$$

$$[a_{t+1}]: \quad -\lambda_t(1+n) + \lambda_{t+1}(1+r_{t+1}) = 0 \tag{1.62}$$

$$[\lambda_t]: \quad (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1} = 0 \tag{1.63}$$

Combining equations (1.61) and (1.62), we obtain a non-arbitrage condition analogous to that of Section 1.1:

$$R_t - \delta = r_t$$

Similarly, the Euler equation is:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1-\delta+R_{t+1})}{(1+n)} \tag{1.64}$$

The following are sufficient and necessary conditions for an allocation to solve the household's problem:

Euler eq.:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1-\delta+R_{t+1})}{(1+n)}, \quad \forall t \geq 0 \tag{1.65}$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1+r_{t+1})}{(1+n)}, \quad \forall t \geq 0 \tag{1.66}$$

Bud. const.:

$$c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \quad \forall t \geq 0 \tag{1.67}$$

Transv. cond.:

$$\lim_{t\to\infty} \beta^t u'(c_t) k_{t+1} = 0 \tag{1.68}$$

$$\lim_{t\to\infty} \beta^t u'(c_t) a_{t+1} = 0 \tag{1.69}$$

Init. cond.:

$$k_0, a_0 \text{ given} \tag{1.70}$$

For the firm's problem, we must be careful in the following: as opposed to the household, which chooses per-capita variables, the firms choose absolute capital and labor. For this

reason, we must solve the firms' problem in absolute terms, and then express everything in per-capita terms.

The first order conditions for the firm yield:

$$w_t = F_l(K_t, L_t)$$
$$R_t = F_k(K_t, L_t)$$

Now, we would like to express the firm's optimality conditions in terms of per-capita variables. In this way, the whole model would be expressed in per-capita terms, so we can easily solve for the equilibrium. For this, note that, given that the production function F satisfies the constant-returns-to-scale condition, we can rewrite $F(K, L)$ as:

$$F(K, L) = F(K \cdot \frac{L}{L}, L) = F(L \cdot \frac{K}{L}, L) = F(L \cdot k, L) = L \cdot F(k, 1)$$

This means that:

$$F_K(K, L) = L \cdot \underbrace{F_k\left(\frac{K}{L}, 1\right) \cdot \frac{1}{L}}_{\text{Chain rule}} = F_k(k, 1) \tag{1.71}$$

$$F_l(K, L) = F(k, 1) + L \cdot \underbrace{F_k\left(\frac{K}{L}, 1\right) \cdot \left(\frac{-K}{L^2}\right)}_{\text{Chain rule}} = F(k, 1) - F_k\left(\frac{K}{L}, 1\right) \cdot \frac{K}{L} = F(k, 1) - F_k\left(k, 1\right) \cdot k$$

In equilibrium, the firm will hire labor L_t and rent capital K_t such that $k_t = K_t/L_t$ satisfies:

$$w_t = F(k_t, 1) - F_k\left(k_t, 1\right) \cdot k_t \tag{1.72}$$
$$R_t = F_k(k_t, 1) \tag{1.73}$$

In equilibrium, given wages (1.72) and rental prices (1.73), the firm makes zero profits, so

per-capita profits are $\pi_t = 0$:

$$\begin{aligned}
\pi_t &= \frac{F(K_t, L_t)}{L_t} - \frac{w_t L_t}{L_t} - \frac{r_t K_t}{L_t} = F(k_t, 1) - w_t - R_t k_t \\
&= F(k_t, 1) - \underbrace{(F(k_t, 1) - F_k(k_t, 1) \cdot k_t)}_{w_t} - \underbrace{F_k(k_t, 1)}_{R_t} k_t \\
&= F(k_t, 1) - F(k_t, 1) + F_k(k_t, 1) \cdot k_t - F_k(k_t, 1) \cdot k_t = 0 \qquad (1.74)
\end{aligned}$$

In this way, the equilibrium is fully characterized by prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, and allocations $\{c_t, k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$, that satisfy equations (1.65)-(1.70) of the household, (1.72) and (1.73) of the firm, and the market clearing conditions.

Using the solution to the firm's problem, it can be shown that the allocations of the social planner are equivalent to those of the competitive equilibrium. This happens because in this context the welfare theorems continue to hold.

1.2.3 Steady State

A steady state is a competitive equilibrium in which all variables per capita remain constant over time. In Section 1.1.3 this was trivial, since per-capita variables were equal to aggregate variables, so there was no growth at all. In this section, in steady state per-capita variables remain constant, but aggregate variables grow.

It is straightforward to verify that capital in steady state is uniquely pinned down by:

$$\frac{1}{\beta} = \frac{(1 - \delta + F_k(k^*, 1))}{(1 + n)}$$

Once capital per capita k^* is determined, all other variables can be pinned down using the optimality conditions:

$$\begin{aligned}
c^* &= F(k^*, 1) - (\delta + n)k^* \\
i^* &= \delta k^* \\
w^* &= F(k^*, 1) - F_k(k^*, 1)k^* \\
R^* &= F_k(k^*, 1) \\
r^* &= F_k(k^*, 1) - \delta
\end{aligned}$$

Now, even though in steady state per-capita variables are constant, aggregate variables are not. Recall that $k_t = K_t/L_t$, $c_t = C_t/L_t, \ldots$, which means that in steady state:

$$\begin{aligned} K_t &= k^* L_t = (1+n)^t k^* \\ C_t &= c^* L_t = (1+n)^t c^* \\ &\vdots \end{aligned}$$

In steady state aggregate variables grow at a rate n, the same as population. This model is the first step towards models with positive growth.

1.3 Endogenous labor supply

In Sections 1.1 and 1.2 we assumed that the utility function of households is a function only of consumption and did not depend on leisure. This implies that households allocate all of their time to work and nothing to leisure. This may not be a realistic assumption, given that households decide every day how much time to devote to work and how much to leisure. In particular, the model without labor decisions is not useful to study topics such as labor markets, taxation, human capital, social security systems, among others. In this section we extend the model presented to include endogenous labor decisions.

Assume again that the economy is composed of representative households and firms with the same characteristics as before. In particular, let's assume that households have size L_t and each individual has 1 unit of time available to devote to labor or leisure, so the total amount of time available within the household equals exactly L_t. Household size grows at a constant rate n, such that $L_{t+1} = (1+n)L_t$.

However, assume that now households obtain utility from the fraction of time devoted to leisure by all of its members. This means that households do not necessarily supply inelastically all of their time to labor, but rather choose how much time to work out of the total available time. If we denote by L_t^s the total amount of time devoted to work by all members of a household of size L_t, the total time devoted to leisure is $L_t - L_t^s$. Therefore, the utility function is now:

$$\sum_{t=0}^{\infty} \beta^t u\left(\frac{C_t}{L_t}, \frac{L_t - L_t^s}{L_t}\right)$$

Here, we assume that the utility function is concave in both arguments and the Inada conditions on consumption hold:

1. $u_c, u_l > 0$
2. $u_{cc}, u_{ll} < 0$
3. $u_{cl} > 0$
4. $\lim_{c\to 0} u_c(c,l) = \infty$
5. $\lim_{c\to\infty} u_c(c,l) = 0$

The budget constraint that the household faces every period is:

$$C_t + K_{t+1} + A_{t+1} = (1-\delta)K_t + w_t L_t^s + R_t K_t + (1+r_t)A_t$$

where now $L_t^s \leq L_t$. Again, I omitted including the profits that the household receives for owning the firm, as in equilibrium they turn out to be equal to zero. The problem of the household can be written in per-capita terms as:

$$\max_{c_t, k_{t+1}, l_t^s} \quad \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - l_t^s\right) \quad s.t. \tag{1.75}$$

$$\forall t \in \{0, 1, \ldots\}: \quad c_t + (1+n)k_{t+1} + (1+n)a_{t+1} = (1-\delta)k_t + w_t l_t^s + R_t k_t + (1+r_t)a_t$$

$$c_t, k_{t+1} \geq 0, \quad 0 \leq l_t^s \leq 1, \quad k_0 \text{ given}, \quad a_0 = 0 \text{ given}$$

$$\lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t}\right) a_{T+1} = 0$$

Now, the average fraction of time devoted to labor within the household l_t^s is a choice. Moreover, there is an additional restriction that bounds the total fraction of time devoted to labor: $0 \leq l_t^s \leq 1$. This means that households cannot work more than the total amount of time available, equal to 1.

In this section, I omit the definition of a competitive equilibrium, as it is analogous to Definition 2. The only difference is that households now choose labor supply l_t^s, so the labor market clearing condition becomes $l_t^s = l_t^d$.

The lagragean associated to the household's problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \beta^t u\left(c_t, 1 - l_t^s\right) + \sum_{t=0}^{\infty} \lambda_t \left((1-\delta)k_t + w_t l_t^s + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1}\right)$$

The first order conditions are:

$$[c_t]: \quad \beta^t u_c(c_t, 1 - l_t^s) - \lambda_t = 0$$

$$[l_t]: \quad -\beta^t u_l(c_t, 1 - l_t^s) + \lambda_t w_t = 0$$

$$[k_{t+1}]: \quad -\lambda_t(1+n) + \lambda_{t+1}\left(1 - \delta + R_{t+1}\right) = 0$$

$$[a_{t+1}]: \quad -\lambda_t(1+n) + \lambda_{t+1}\left(1 + r_{t+1}\right) = 0$$

$$[\lambda_t]: \quad (1-\delta)k_t + w_t l_t^s + R_t k_t + (1+r_t)a_t - c_t - (1+n)k_{t+1} - (1+n)a_{t+1} = 0$$

With endogenous labor choice, we have an additional first order condition, associated with the optimal choice of labor supply. Combining the first two conditions:

$$w_t \cdot u_c(c_t, 1 - l_t^s) = u_l(c_t, 1 - l_t)$$

This equation is often called the **intra-temportal** optimality condition. In the optimum, the individual is indifferent between having an additional unit of leisure, or working an additional unit of time and consuming the income. The left hand side of this equation represents the marginal utility of working an additional unit of time: an additional unit of time will generate income w_t to the household that can be spent in consumption, whose marginal utility is $u_c(c_t, 1 - l_t)$. The right hand side is the marginal utility of leisure. In equilibrium, the marginal gains of working or enjoying leisure are the same.

As in past sections, the other optimality conditions are the Euler equations associated to capital and risk-free assets, the budget constraint, and the transversality conditions. The optimal allocation $\{c_t, l_t^s, k_{t+1}, a_{t+1}\}_{t=0}^{\infty}$ for the consumer, given prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, is fully characterized by:

Euler eq.:
$$\frac{u_c(c_t, 1 - l_t^s)}{\beta u_c(c_{t+1}, 1 - l_{t+1}^s)} = \frac{(1 - \delta + R_{t+1})}{(1 + n)}, \quad \forall t \geq 0 \tag{1.76}$$

$$\frac{u_c(c_t, 1 - l_t^s)}{\beta u_c(c_{t+1}, 1 - l_{t+1}^s)} = \frac{(1 + r_{t+1})}{(1 + n)}, \quad \forall t \geq 0 \tag{1.77}$$

Intratemporal:
$$u_c(c_t, 1 - l_t^s) = \frac{u_l(c_t, 1 - l_t)}{w_t}, \quad \forall t \geq 0 \tag{1.78}$$

Bud. const.:
$$c_t + (1 + n)k_{t+1} + (1 + n)a_{t+1} = (1 - \delta)k_t + w_t l_t^s + R_t k_t + (1 + r_t)a_t \quad \forall t \geq 0 \tag{1.79}$$

Transv. cond.:
$$\lim_{t \to \infty} \beta^t u_c(c_t, 1 - l_t^s) k_{t+1} = 0 \tag{1.80}$$

$$\lim_{t \to \infty} \beta^t u_c(c_t, 1 - l_t^s) a_{t+1} = 0 \tag{1.81}$$

Init. cond.:
$$k_0, a_0 \text{ given} \tag{1.82}$$

The firm's problem is analogous to that of past sections, so I omit it in this section. In particular, equilibrium factor prices w_t and R_t are pinned down by equations (1.72) and

(1.73).

1.3.1 Steady State

It can be verified that in the stationary steady state the values of c^*, y^*, k^* and l^* are uniquely pinned down by the following equations:

$$\frac{1}{\beta} = \frac{(1 - \delta + F_K(k^*, l^*))}{(1 + n)}$$

$$\cdot u_c(c^*, 1 - l^*) = \frac{u_l(c^*, 1 - l^*)}{F_L(k^*, l^*)}$$

$$c^* = F(k^*, l^*) - (n + \delta)k^*$$

$$\vdots$$

1.4 Exogenous Growth and Technological Change

Up to now, the model does not allow for long term growth; in steady state absolute variables grow at exactly the same rate as population, so per capita variables remain constant. This is not what we observe in reality. Most economies experience growth beyond the increase in population size. This section describes a model in which there is exogenous growth generated in steady state. Section 2 will introduce models in which long-term growth is endogenously generated.

In this section, for simplicity, we will again assume that leisure does not generate utility to households and labor is inelastically supplied. The model can be extended to have endogenous labor supply as in Section 1.3.

Let's assume now that the production function has a technological parameter A_t that augments labor productivity $F(K_t, A_t L_t)$, and grows at an exogenous rate g, such that:

$$A_{t+1} = (1+g)A_t$$

If we assume that productivity in period 0 is $A_0 = 1$, this means that $A_t = (1+g)^t$.

Note that, naturally, our economy will never reach a steady state, as productivity is growing in the long run, so the economy will be capable of increasing production, consumption and capital per worker period after period. Instead, we will study the economy when it reaches a state in which per-capita variables *grow at a constant rate*, which is commonly known as a **balanced growth path**. Note that a steady state, which is when all per-capita variables remain constant, is a particular case of a balanced growth path, in which variables grow at a rate exactly equal to zero.

Before solving the model, let's understand what a balanced growth path implies, and analyze the conditions under which a balanced growth path can exist. First, on a balanced growth path all variables grow at a constant rate. In principle, growth rates need to be the same for different variables. However, as the next theorem shows, on a balanced growth path all variables grow at the same rate.

Theorem 3. *Along a balanced growth path, all per-capita variables grow at the same rate.*

Proof. Assume we solve the model in per-capita terms, as in Section 1.2 with population growth. The resulting resource constraint is one of the necessary and sufficient conditions

for an optimum:

$$c_t = F(k_t, A_t) + (1-\delta)k_t - (1+n)k_{t+1} \tag{1.83}$$

Define the growth rate of per-capita consumption along a balanced growth path as γ_c, such that $c_t = (1+\gamma_c)^t c_0$, assuming the economy starts at period 0 on the balanced growth path. Similarly, define γ_k and γ_y as the growth rates of per-capita capital and income on a balanced growth path, such that $k_t = (1+\gamma_k)^t k_0$ and $y_t = (1+\gamma_y)^t y_0$.

Using these definitions, we can express the resource constraint (1.83) as:

$$(1+\gamma_c)^t c_0 = F((1+\gamma_k)^t k_0, (1+g)^t) + (1-\delta)(1+\gamma_k)^t k_0 - (1+n)(1+\gamma_k)^{t+1} k_0$$

Dividing both sides by $(1+\gamma_k)^t$:

$$\left(\frac{1+\gamma_c}{1+\gamma_k}\right)^t c_0 = F\left(k_0, \left(\frac{1+g}{1+\gamma_k}\right)^t\right) + (1-\delta)k_0 - (1+n)(1+\gamma_k)k_0$$

The only way for this equation to hold for all t is if the terms that depend on t do not vary, which implies that:

$$\left(\frac{1+\gamma_c}{1+\gamma_k}\right) = \left(\frac{1+g}{1+\gamma_k}\right) = 1$$

This implies that $\gamma_c = \gamma_k = g$. Also, output per-capita is:

$$y_t = F((1+g)^t k_0, (1+g)^t) = (1+g)^t F(k_0, 1) = (1+\gamma_y)^t y_0$$

This implies that $\gamma_y = \gamma_c = \gamma_k = g$. □

Re-writing the resource constraint:

$$(1+\gamma_y)^t y_0 = (1+\gamma_c)^t c_0 + (1+n)(1+\gamma_k)^{t+1} k_0 - (1-\delta)(1+\gamma_k)^t k_0$$

$$\iff \quad y_0 = c_0 + (1+n)(1+g)k_0 - (1-\delta)k_0$$

$$\iff \quad y_0 = c_0 + \underbrace{(n+g+ng+\delta)k_0}_{\text{Investment}}$$

This means that, as long as population and technological growth are small (say, around $1-2\%$, which means $ng \approx 0.0002 \approx 0$), investment in the balanced growth path replaces depreciated capital (δk_0), and augments the capital stock such that capital per capita and per level of technology remains constant next period ($(n+g)k_0$).

Now, let's analyze the conditions under which a balanced growth path can exist. Take the Euler equation resulting from solving the model in per-capita terms:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{1-\delta+F_k(k_{t+1},A_{t+1})}{(1+n)} \tag{1.84}$$

On the right-hand side, the only non-constant term is $F_k(k_{t+1},A_{t+1})$. Because of the constant-returns-to-scale assumption, $F_k(k_{t+1},A_{t+1}) = F_k(k_{t+1}/A_{t+1},1)$. On a balanced growth path, given that $k_t = (1+g)^t k_0$ and $A_t = (1+g)^t$, the term $k_{t+1}/A_{t+1} = k_0$ remains constant. Therefore, on a balanced growth path, the term $u'(c_t)/\beta u'(c_{t+1})$ must be constant. This implies the following result:

Theorem 4. *There exists a balanced growth path if and only if u is CRRA.*

Proof. ($\Longleftarrow$) :Assume the utility function u is CRRA, which means that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. This means that $u'(c) = c^{-\sigma}$, and the Euler equation (1.84) becomes:

$$\frac{1}{\beta}\cdot\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} = \frac{1-\delta+F_k(k_0,1)}{(1+n)}$$

On a balanced growth path, $c_{t+1} = (1+g)c_t$, so there exists a balanced growth path as long as the economy starts at a capital level k_0, such that:

$$\frac{1}{\beta}\cdot(1+g)^{\sigma} = \frac{1-\delta+F_k(k_0,1)}{(1+n)}$$

($\Longrightarrow$) : Now, assume there exists a balanced growth path, such that $c_t = (1+g)c_{t+1}$. Given the Euler equation (1.84), this means that:

$$\frac{u'(c_t)}{\beta u'((1+g)c_t)} = M$$

For some constant M. Rearranging and differentiating with respect to c_t:

$$
\begin{aligned}
& u'(c_t) = M \cdot \beta u'((1+g)c_t) \\
\iff \quad & u''(c_t) = M \cdot \beta(1+g)u''((1+g)c_t) \\
\iff \quad & u''(c_t) = (1+g)\frac{u'(c_t)}{\beta u'((1+g)c_t)} \cdot \beta u''((1+g)c_t) \\
\iff \quad & \frac{u''(c_t)c_t}{u'(c_t)} = \frac{u''((1+g)c_t)(1+g)c_t}{u'((1+g)c_t)} = \frac{u''(c_{t+1})c_{t+1}}{u'(c_{t+1})}
\end{aligned}
$$

This means that $\frac{u''(c_t)c_t}{u'(c_t)}$ is constant, which is the definition of CRRA. □

Therefore, the only models in which a balanced growth path can exist are those in which the utility function takes the CRRA form. Moreover, on a balanced growth path all variables grow at the same rate as the technological change.

1.4.1 Solving for a Competitive Equilibrium

This section solves for the competitive equilibrium when there is technological change. As before, I omit the definition of a competitive equilibrium, given that it is analogous to the definitions in past sections.

Note that we can re-state the definition by saying that the economy reaches a balanced growth path, whenever all variables *per effective unit of labor* are constant. Henceforth, we will denote variables per effective unit of labor with a tilde:

$$
\begin{aligned}
\tilde{c}_t &= \frac{C_t}{A_t L_t} = \frac{c_t}{A_t} \\
\tilde{y}_t &= \frac{Y_t}{A_t L_t} = \frac{y_t}{A_t} \\
\tilde{k}_t &= \frac{K_t}{A_t L_t} = \frac{k_t}{A_t} \\
&\vdots
\end{aligned}
$$

We now write the household's problem in terms of per-effective-unit-of-labor variables. The

household's budget constraint becomes:

$$\tilde{c}_t + \tilde{i}_t + (1+g)(1+n)\tilde{a}_{t+1} = \tilde{w}_t + r_t\tilde{k}_t + \tilde{\pi}_t + \tilde{a}_t$$

where $\tilde{w}_t = w_t/A_t$. Now, to express the capital evolution equation in terms of variables per efficiency unit, note that:

$$\frac{K_{t+1}}{A_t L_t} = \frac{K_{t+1}}{A_t L_t}\left(\frac{A_{t+1}L_{t+1}}{A_{t+1}L_{t+1}}\right) = \frac{K_{t+1}}{A_{t+1}L_{t+1}}\left(\frac{A_{t+1}}{A_t}\right)\left(\frac{L_{t+1}}{L_t}\right) = (1+g)(1+n)\tilde{k}_{t+1}$$

This means that the capital evolution equation can be written as:

$$(1+g)(1+n)\tilde{k}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{i}_t$$

The total output in terms of per-effective-units-of-labor is:

$$\tilde{y}_t = \frac{F(K_t, A_t L_t)}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, \frac{A_t L_t}{A_t L_t}\right) = F(\tilde{k}_t, 1)$$

Now, we explicitly use the CRRA utility function, as stated in the previous theorem:

$$u(c) = \frac{c^{1-\sigma}}{1-\sigma}$$

If we multiply and divide by A_t, we get:

$$\begin{aligned}
\sum_{t=0}^{\infty} \beta^t u\left(c_t\right) &= \sum_{t=0}^{\infty} \beta^t \frac{\left(\frac{A_t}{A_t} c_t\right)^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty} \beta^t \frac{\left(A_t \tilde{c}_t\right)^{1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty} \beta^t A_t^{1-\sigma} \cdot \frac{\tilde{c}_t^{\,1-\sigma}}{1-\sigma} \\
&= \sum_{t=0}^{\infty} \hat{\beta}^t \frac{\tilde{c}_t^{\,1-\sigma}}{1-\sigma}
\end{aligned}$$

where $\hat{\beta} = \beta \cdot (1+g)^{1-\sigma}$. The household's problem is given by:

$$\max_{\tilde{c}_t, \tilde{k}_{t+1}} \quad \sum_{t=0}^{\infty} \hat{\beta}^t u\left(\tilde{c}_t\right) \quad s.t. \tag{1.85}$$

$$\forall t \in \{0, 1, \ldots\}: \quad \tilde{c}_t + (1+g)(1+n)\tilde{k}_{t+1} + (1+g)(1+n)\tilde{a}_{t+1} = (1-\delta)\tilde{k}_t + \tilde{w}_t + R_t\tilde{k}_t + (1+r)\tilde{a}_t$$

$$\tilde{c}_t, \tilde{k}_{t+1} \geq 0, \quad \tilde{k}_0 \text{ given}$$

$$\lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t}\right) \tilde{a}_{T+1} \geq 0$$

The solution to the household's problem yields equations that are analogous to past sections'. In particular, the solution yields the following Euler equation:

$$\frac{u'(\tilde{c}_t)}{\hat{\beta}u'(\tilde{c}_{t+1})} = \frac{(1-\delta+R_{t+1})}{(1+n)(1+g)}$$

The firm's problem is:

$$\max_{L_t, K_t} \quad F(K_t, A_tL_t) - w_tL_t - r_tK_t$$

So prices are set by:

$$\begin{aligned} w_t &= A_t \cdot F_l(K_t, A_tL_t) \\ r_t &= F_k(K_t, A_tL_t) \end{aligned}$$

To express prices in per-efficiency units of labor:

$$F(K_t, A_tL_t) = A_tL_tF\left(\frac{K_t}{A_tL_t}, 1\right) = A_tL_tF(\tilde{k}_t, 1)$$

So:

$$\begin{aligned} A_tF_l(K_t, A_tL_t) &= A_t \cdot \left[F\left(\frac{K_t}{A_tL_t}, 1\right) + L_tF_k\left(\frac{K_t}{A_tL_t}, 1\right)\left(\frac{-K_t}{A_tL_t^2}\right)\right] \\ &= A_t\left[F\left(\frac{K_t}{A_tL_t}, 1\right) + F_k\left(\frac{K_t}{A_tL_t}, 1\right)\left(\frac{-K_t}{A_tL_t}\right)\right] \\ &= A_tF(\tilde{k}_t, 1) + A_tF_k(\tilde{k}_t, 1)\tilde{k}_t \end{aligned}$$

Therefore:

$$\begin{aligned}\tilde{w}_t &= F(\tilde{k}_t, 1) + F_k(\tilde{k}_t, 1)\tilde{k}_t \\ r_t &= F_k(\tilde{k}_t, 1)\end{aligned}$$

1.4.2 Balanced Growth Path

On a balanced growth path, all variables per effective unit of labor remain constant:

$$\begin{aligned}\tilde{c}_t &= \tilde{c_{t+1}} = \ldots = \tilde{c}^* \\ \tilde{k}_t &= \tilde{k_{t+1}} = \ldots = \tilde{k}^* \\ &\vdots\end{aligned}$$

This implies that, on a balanced growth path:

$$F_K(\tilde{k}^*, 1) = \frac{(1+n)(1+g)}{\hat{\beta}} - 1 + \delta \tag{1.86}$$

$$\tilde{c}^* = F(\tilde{k}^*, 1) - (\delta + n + g + ng)\tilde{k}^* \tag{1.87}$$

Once we have solved for the balanced growth path variables, we can obtain the per-capita variables by multiplying by $A_t = (1+g)^t$:

$$k_t = (1+g)^t k^*, \quad c_t = (1+g)^t c^*, \quad \ldots$$

If we plot $\log(y_t), \log(k_t), \ldots$ against time t, we obtain a linear slope, as is the case for many developed countries in the last century. Figure 1.1 illustrates per-capita GDP over the last century, where the vertical axis represents constant U.S. dollars in logarithmic scale. According to this Figure, the neoclassical growth model seems a sufficiently good tool to analyze growth in economies that seem to have reached a "balanced growth path", such as the U.S. or some European countries.

The main advantage of this section, as opposed to past sections, is that with exogenous growth we have a model in which per-capita variables grow in the long run, as is observed for most nations. However, so far we have assumed that growth is driven by growth in technology, which follows a completely exogenous process. Later sections will study models in which technological progress is an endogenous variable in the model.

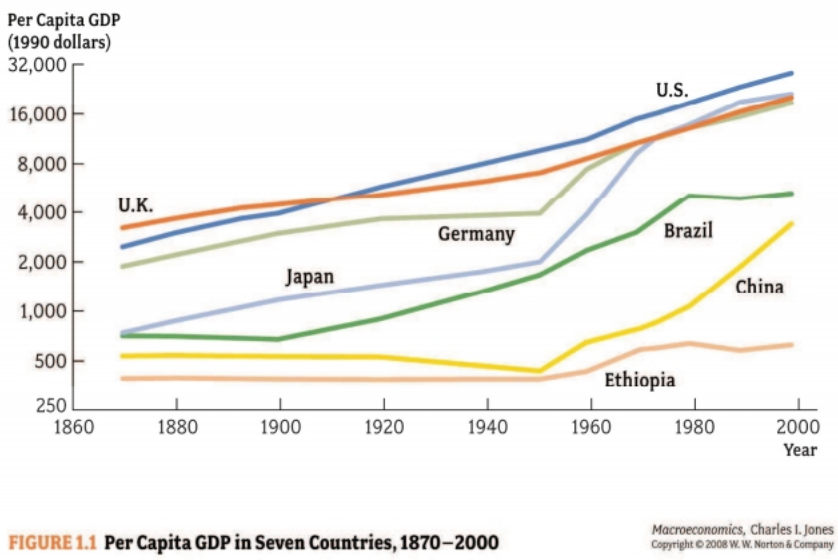


Figure 1.1: Per-capita GDP.

1.5 Numerical Methods I

Up to now, we have studied economies in which the equilibrium and/or steady state is fully characterized by a set of equations but, in most cases, there is no closed-form solution for each of the variables. This section allows us to compute the actual values of the variables using a method to solve non-linear equations called the Newton-Raphson method. The importance of this method is illustrated in the following two examples:

1.5.1 Example I - Computation of Steady State/Balanced Growth Path

Recall that in past sections we computed the steady state/balanced growth path values of capital k^*, consumption c^*, production y^*, and prices w^*, R^* and r^* in the economy (see equations (1.42)-(1.47)). To obtain consumption and production, we first had to solve for capital in steady state/balanced growth path, implicitly defined by equation (1.42). The question that arises is, given parameters for the models described (β, n, g, ...), how can we numerically obtain the value of k^*?

For example, note that for the model with technological change (equation (1.86)), the level of capital per-efficiency unit of labor in steady state is given by the value $\tilde{k}^*$ that

solves:

$$F_k(\tilde{k}^*, 1) = \frac{(1+n)(1+g)}{\hat{\beta}} - 1 + \delta$$

This equation can be rearranged, so that $\tilde{k}^*$ solves:

$$F_k(\tilde{k}^*, 1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta = 0$$

So the capital per-efficiency unit of labor is determined by the zero, or root, of the non-linear equation:

$$g(x) = F_k(x, 1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta$$

That is, $\tilde{k}^*$ is such that $g(\tilde{k}^*) = 0$. The Newton-Raphson method studied in this section is a first approach to finding the zero of this equation, which will allow us to compute the values of the variables in steady state.

1.5.2 Example II - Computation of Equilibrium

The second example to illustrate the importance of solving systems of non-linear equations is a simplified two-period version of the neoclassical growth model. Assume that the individual only lives for two periods, and chooses consumption every period and the amount of capital to accumulate from the first to the second period. Further, assume there is no population growth. The problem of the household is:

$$\max_{c_0, c_1, k_1} \quad u(c_0) + \beta u(c_1) \quad s.t.$$

$$c_0 + k_1 = (1-\delta)k_0 + w_0 + r_0 k_0$$

$$c_1 = (1-\delta)k_1 + w_1 + r_1 k_1$$

$$c_0, c_1, k_1 \geq 0, \quad k_0 \text{ given}$$

The lagrangean is:

$$\mathcal{L} = u(c_0) + \beta u(c_1) + \lambda_0 ((1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1) + \lambda_1 ((1-\delta)k_1 + w_1 + r_1 k_1 - c_1)$$

The first order conditions are:

$$
\begin{aligned}
[c_0]: &\quad u_c(c_0) - \lambda_0 = 0 \\
[c_1]: &\quad \beta u_c(c_1) - \lambda_1 = 0 \\
[k_1]: &\quad -\lambda_0 + \lambda_1 (1 - \delta + r_1) = 0 \\
[\lambda_0]: &\quad (1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 = 0 \\
[\lambda_1]: &\quad (1-\delta)k_1 + w_1 + r_1 k_1 - c_1 = 0
\end{aligned}
$$

Which, getting rid of the λ's, yields the following optimality conditions:

$$u_c(c_0) = (1 - \delta + r_1)\, \beta u_c(c_1) \tag{1.88}$$

$$(1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 = 0 \tag{1.89}$$

$$(1-\delta)k_1 + w_1 + r_1 k_1 - c_1 = 0 \tag{1.90}$$

The first order conditions for the firm are:

$$w_0 = F_l(k_0, 1) \tag{1.91}$$

$$w_1 = F_l(k_1, 1) \tag{1.92}$$

$$r_0 = F_k(k_0, 1) \tag{1.93}$$

$$r_1 = F_k(k_1, 1) \tag{1.94}$$

In order to compute the equilibrium of the economy described we must find $(c_0, c_1, k_1, w_0, w_1, r_0, r_1)$ that solve the system of equations (1.88)-(1.94). We can rearrange each equation, to define the function f in the following way:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \end{pmatrix} = \begin{pmatrix} u_c(c_0) - (1 - \delta + r_1)\, \beta u_c(c_1) \\ (1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 \\ (1-\delta)k_1 + w_1 + r_1 k_1 - c_1 \\ w_0 - F_l(k_0, 1) \\ w_1 - F_l(k_1, 1) \\ r_0 - F_k(k_0, 1) \\ r_1 - F_k(k_1, 1) \end{pmatrix}$$

Where $x = (c_0, c_1, k_1, w_0, w_1, r_0, r_1)$. Finding an equilibrium to such economy is equivalent to finding a root, or zero, of the function f. That is, an equilibrium is x^* such that $f(x^*) = 0$. We can also get rid of the last 4 equations, by plugging in the prices directly in the optimality conditions of the household:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \end{pmatrix} = \begin{pmatrix} u_c(c_0) - (1 - \delta + F_k(k_1, 1))\, \beta u_c(c_1) \\ (1-\delta)k_0 + F_l(k_0, 1) + F_k(k_0, 1)k_0 - c_0 - k_1 \\ (1-\delta)k_1 + F_l(k_1, 1) + F_k(k_1, 1)k_1 - c_1 \end{pmatrix}$$

As in last example, the Newton-Raphson will be useful to solve for the equilibrium of the economy.

1.5.3 Newton-Raphson

The reasoning behind the method is as follows. Assume we have a vector $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, and a function $f : \mathbb{R}^n \to \mathbb{R}^n$, which can be written as:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_n(x) \end{pmatrix}$$

Suppose we are interested in finding a root to that function. That is, suppose we are interested in finding $\bar{x} \in \mathbb{R}^n$, such that $f(\bar{x}) = 0$, and assume we start from a point $\hat{x}$ which is sufficiently close to $\bar{x}$. Consider the following Taylor expansion of the function f around the point $\bar{x}$:

$$f(\bar{x}) \approx f(\hat{x}) + J(\hat{x}) \cdot (\bar{x} - \hat{x})$$

Where J is the Jacobian, or matrix of first derivatives of f. Given that $f(\bar{x}) = 0$, we can rewrite it as:

$$0 \approx f(\hat{x}) + J(\hat{x}) \cdot (\bar{x} - \hat{x}) \quad \Longleftrightarrow \quad \bar{x} \approx \hat{x} - J^{-1}(\hat{x}) \cdot f(\hat{x}) \tag{1.95}$$

This means that, if we start from a guess $\hat{x}$ of the root of f that is sufficiently close to $\bar{x}$, we can approximate $\bar{x}$ using equation (1.95).

In matrix form, this can be written as:

$$\begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{pmatrix} \approx \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \vdots \\ \hat{x}_n \end{pmatrix} - \begin{pmatrix} \frac{\partial f_1(\hat{x})}{\partial x_1} & \dots & \frac{\partial f_1(\hat{x})}{\partial x_n} \\ \frac{\partial f_2(\hat{x})}{\partial x_1} & \dots & \frac{\partial f_2(\hat{x})}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n(\hat{x})}{\partial x_1} & \dots & \frac{\partial f_n(\hat{x})}{\partial x_n} \end{pmatrix}^{-1} \cdot \begin{pmatrix} f_1(\hat{x}) \\ f_2(\hat{x}) \\ \vdots \\ f_n(\hat{x}) \end{pmatrix}$$

Therefore, the Newton-Raphson algorithm is as follows:

Algorithm 1 (Newton-Raphson)**.**

1. *Set $s = 0$, and start with a guess x^0 for the root of f.*

2. *Compute the function $f(x^s)$ and the Jacobian $J(x^s)$.*

3. *Set x^{s+1} as follows:*

$$x^{s+1} = x^s - J(x^s)^{-1} f(x^s)$$

4. *If $\left\| x^{s+1} - x^s \right\| < \epsilon$, stop. Else, set $s = s + 1$ and go to 2.*

The algorithm is illustrated in Figure 1.2[1]. In this algorithm, the norm $\|\cdot\|$ can be the euclidean norm, or any other norm in $\mathbb{R}^n$. If the starting point x^0 is sufficiently close to the root $\bar{x}$, it can be shown that the method converges and the result obtained is close to the root. However, if we start the method far from the root, the algorithm can converge to another root, or diverge.

With this algorithm, we can approximate numerically the roots to non-linear systems of equations. How can we apply the method to solve for the steady state values in Section 1.5.1, and for the equilibrium in Section 1.5.2?

[1]Source: https://www.geeksforgeeks.org/program-for-newton-raphson-method/

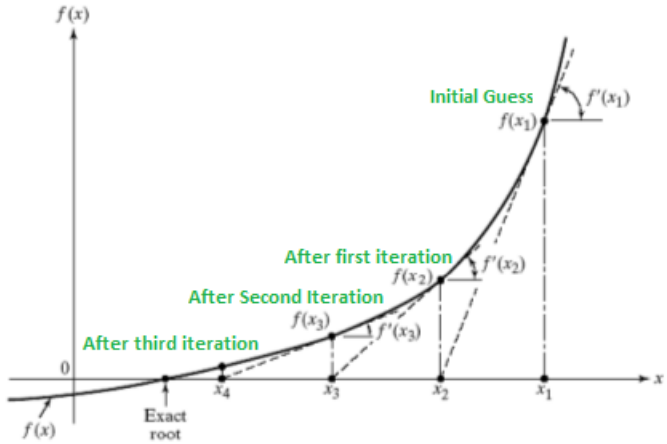


Figure 1.2: Illustration of Newton-Raphson method.

Example I - Computation of Steady State

Recall that to find the level of capital in steady state, it suffices to find the value x such that $g(x) = 0$, where:

$$g(x) = F_k(x, 1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta$$

The Jacobian of g is simply its derivative with respect to x, equal to:

$$J(x) = F_{kk}(x, 1)$$

Therefore, in every iteration the algorithm updates the value of x as:

$$x^{s+1} = x^s - F_{kk}(x^s, 1) \cdot \left(F_k(x^s, 1) - \frac{(1+n)(1+g)}{\hat{\beta}} + 1 - \delta \right)$$

If we start at a point x^0 which is sufficiently close to the value $\tilde{k}^*$, the Newton-Raphson algorithm will converge to an approximation of $\tilde{k}^*$.

This algorithm can be implemented in the way described above, by computing the derivative and iterating until convergence. However, most programming languages already have built-in functions that solve non-linear systems of equations. In `Matlab`, a function to solve non-linear equations is `fsolve`, which receives as input a function g and an initial

guess of a root x_0. For example, if we assume that the production function takes the form of a Cobb-Douglas function, such that $F(K, L) = K^{\alpha}L^{1-\alpha}$ and $F_K(k, 1) = \alpha k^{\alpha-1}$, the code in Box 1.1 computes the steady state value of capital.

```
% Parameter values
aalpha = 0.3;
bbeta  = 0.95;
n      = 0.02;
g      = 0.03;

% Defining the function g
g = @(x) aalpha*(x^(aalpha-1))-((1+n)*(1+g))/bbeta+1-ddelta
x0 = 1;

% Non-linear equation solver
fsolve(g, x0);
```

Box 1.1: `MATLAB` code to compute steady state level of capital.

Example II - Computation of Equilibrium

Recall that finding an equilibrium in the two-period environment is equivalent to finding a root for f, where:

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ f_3(x) \\ f_4(x) \\ f_5(x) \\ f_6(x) \\ f_7(x) \end{pmatrix} = \begin{pmatrix} u_c(c_0) - (1 - \delta + r_1)\,\beta u_c(c_1) \\ (1-\delta)k_0 + w_0 + r_0 k_0 - c_0 - k_1 \\ (1-\delta)k_1 + w_1 + r_1 k_1 - c_1 \\ w_0 - F_l(k_0, 1) \\ w_1 - F_l(k_1, 1) \\ r_0 - F_k(k_0, 1) \\ r_1 - F_k(k_1, 1) \end{pmatrix}$$

In order to apply the algorithm directly, we can compute the Jacobian matrix by finding all the cross derivatives of f. For example:

$$\begin{aligned} f_{11}(x) &= u_{cc}(c_0) \\ f_{12}(x) &= -(1-\delta+r_1)\beta u_{cc}(c_1) \\ f_{21}(x) &= -1 \\ f_{22}(x) &= 0 \\ &\vdots \end{aligned}$$

As in the past example, we can use the `fsolve` function. Box 1.2 illustrates the definition of function f.

```
function F = sys_equations(c0, c1, k1, w0, w1, r0, r1)
 F(1) = c0^(-ssigma) - (1-ddelta+r1)*bbeta*c1^(-ssigma);
 F(2) = (1-ddelta)*k0+w0+r0*k0-c0-k1;
 F(3) = (1-ddelta)*k1+w1+r1*k1-c1;
 F(4) = w0 - (1-aalpha)*k0^aalpha;
 F(5) = w1 - (1-aalpha)*k1^aalpha;
 F(6) = r0 - aalpha*k0^(aalpha-1);
 F(7) = r1 - aalpha*k1^(aalpha-1);
end
```

Box 1.2: `MATLAB` code to compute the equilibrium.

1.5.4 Transitional Dynamics

The last example was a simplified, two-period version of the economy described in Section 1.1, where households live for an infinite number of periods. Our ultimate goal is to compute the equilibrium in such an economy. It turns out that solving for that equilibrium is not very different from what we have done so far.

Recall that the optimality conditions in Section 1.1 at every period t are given by:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = (1 - \delta + r_{t+1})$$

$$c_t + k_{t+1} = (1 - \delta)k_t + F(k_t, 1)$$

$$w_t = F_l(k_t, l_t)$$

$$r_t = F_k(k_t, l_t)$$

By plugging the prices in the household's optimality conditions, the equilibrium allocations are fully characterized by the following two equations for every t:

$$\Psi_1(c_t, c_{t+1}, k_t, k_{t+1}) = \frac{u'(c_t)}{\beta u'(c_{t+1})} - (1 - \delta + F_k(k_{t+1}, 1)) = 0$$

$$\Psi_2(c_t, c_{t+1}, k_t, k_{t+1}) = (1 - \delta)k_t + F_l(k_t, 1) + F_k(k_t, 1)k_t - c_t - k_{t+1} = 0$$

The problem is that now, as opposed to the second example above, we have an infinite number of periods, which means that to compute the equilibrium we must solve a system of infinitely many equations. The way around this difficulty is the following. Recall that, no matter what is the starting point k_0, we assumed this economy converges to a steady state in the long run, in which all variables are constant. Technically speaking, if the economy starts *out* of steady state, it will never *exactly* reach steady state, but will get arbitrarily close to it. This means that, for a sufficiently large T, we can assume that the economy effectively reaches steady state, which means that $k_t = k^*$ for all $t \geq T$. Therefore, since variables remain constant after T, the system of equations we need to solve will only include the equations up to period T (those corresponding to $t > T$ are redundant). This assumption reduces the number of equations in our system significantly, to a finite number of equations,

namely:

$$\Psi_1(c_0, c_1, k_0, k_1) = 0$$
$$\Psi_2(c_0, c_1, k_0, k_1) = 0$$
$$\Psi_1(c_1, c_2, k_1, k_2) = 0$$
$$\Psi_2(c_1, c_2, k_1, k_2) = 0$$
$$\dots$$
$$\Psi_1(c_{T-1}, c_T, k_{T-1}, k_T) = 0$$
$$\Psi_2(c_{T-1}, c_T, k_{T-1}, k_T) = 0$$

This means that, to solve for the equilibrium variables, we should solve for $\{c_t, k_{t+1}\}_{t=0}^{T}$. In the above system, there are $2 \cdot (T+1)$ unknowns and $2T$ equations. Recall that k_0 is given and $k_T = k^*$, so can effectively solve for an equilibrium by finding the root of the system of equations.

Figure 1.3 illustrates the behavior of an example economy that starts at 10% the capital of the steady state, assuming the utility function is $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ and the production function is $f(k, 1) = \alpha k^{1-\alpha}$. After 10 periods, the economy has already converged to steady state.

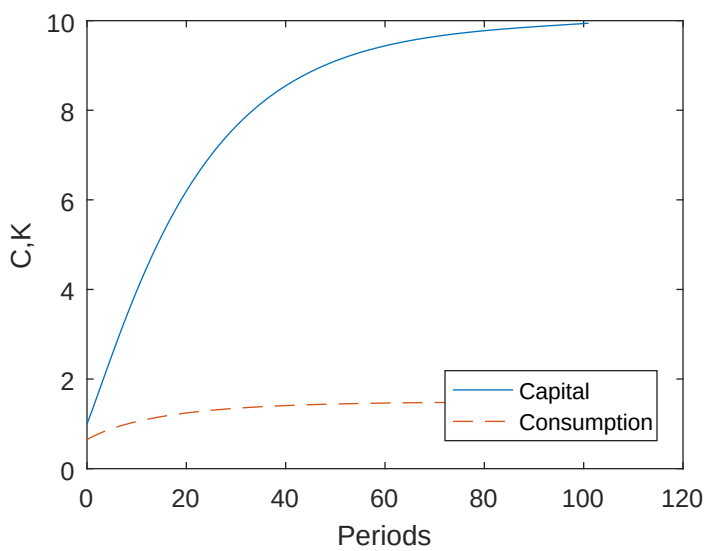


Figure 1.3: Transitional dynamics to steady state.

1.5.5 Approximating the Jacobian

If we want to apply the Newton-Raphson method we must know the Jacobian matrix J of f, which means that we must be able to obtain the derivatives of f with respect to every dimension. In many cases, as the two examples described above, we can obtain the derivatives of f and apply directly Algorithm 1. However, there are cases in which it is not possible to obtain the Jacobian matrix.

If we cannot explicitly compute the derivatives of f, we can obtain a numerical approximation to J. At every iteration, instead of computing $J(x)$, we will have to approximate it.

The Jacobian can be written as:

$$J(x) = [J_1(x), J_2(x), \ldots, J_n(x)]$$

Where $J_i(x)$ is the derivative of f with respect to x_i:

$$J_i(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_i} \\ \frac{\partial f_2(x)}{\partial x_i} \\ \frac{\partial f_3(x)}{\partial x_i} \\ \vdots \\ \frac{\partial f_n(x)}{\partial x_i} \end{pmatrix}$$

Making a Taylor approximation of f, for a sufficiently small $h \geq 0$:

$$\begin{aligned} f(x_1 + h_1, x_2, x_3, \ldots, x_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_1(x)h_1 \\ f(x_1, x_2 + h_2, x_3, \ldots, x_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_2(x)h_2 \\ f(x_1, x_2, x_3 + h_3, \ldots, x_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_3(x)h_3 \\ &\vdots \\ f(x_1, x_2, x_3, \ldots, x_n + h_n) &\approx f(x_1, x_2, x_3, \ldots, x_n) + J_n(x)h_n \end{aligned}$$

Which means that we can approximate $J_1, \ldots, J_n$ as:

$$\begin{aligned}
J_1(x) &\approx \frac{1}{h_1}\left(f(x_1+h_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)\right) \\
J_2(x) &\approx \frac{1}{h_2}\left(f(x_1, x_2+h_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n)\right) \\
&\vdots \\
J_n(x) &\approx \frac{1}{h_n}\left(f(x_1, x_2, x_3, \ldots, x_n+h_n) - f(x_1, x_2, x_3, \ldots, x_n)\right)
\end{aligned}$$

If $h \geq 0$, we are approximating J from the right. Analogously, we can approximate J from the left:

$$\begin{aligned}
J_1(x) &\approx \frac{1}{h_1}\left(f(x_1, x_2, x_3, \ldots, x_n) - f(x_1-h_1, x_2, x_3, \ldots, x_n)\right) \\
J_2(x) &\approx \frac{1}{h_2}\left(f(x_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2-h_2, x_3, \ldots, x_n)\right) \\
&\vdots \\
J_n(x) &\approx \frac{1}{h_n}\left(f(x_1, x_2, x_3, \ldots, x_n) - f(x_1, x_2, x_3, \ldots, x_n-h_n)\right)
\end{aligned}$$

Finally, we can approximate J as an average of the right and left approximations:

$$\begin{aligned}
J_1(x) &\approx \frac{1}{2h_1}\left(f(x_1+h_1, x_2, x_3, \ldots, x_n) - f(x_1-h_1, x_2, x_3, \ldots, x_n)\right) \\
J_2(x) &\approx \frac{1}{2h_2}\left(f(x_1, x_2+h_2, x_3, \ldots, x_n) - f(x_1, x_2-h_2, x_3, \ldots, x_n)\right) \\
&\vdots \\
J_n(x) &\approx \frac{1}{2h_n}\left(f(x_1, x_2, x_3, \ldots, x_n+h_n) - f(x_1, x_2, x_3, \ldots, x_n-h_n)\right)
\end{aligned}$$

In the `MATLAB` examples above, we did not explicitly state whether the algorithm should take a numerical approximation of the Jacobian at every iteration, or use an explicit expression to compute it. Having to approximate J at every iteration might slow down the algorithm, given the additional step needed for this. To avoid this extra cost, we can explicitly tell `MATLAB` that we are going to supply the Jacobian matrix. The code in Box 1.3 activates the option in which we supply the `fsolve` function with the Jacobian matrix.

```
options = optimoptions(options,'SpecifyObjectiveGradient',
    true);
fsolve(@(x) equilibrium(K0, Kss, x), X, options);
```

Box 1.3: `MATLAB` code to supply `fsolve` with Jacobian function.

In order for the `SpecifyObjectiveGradient` to work, the objective function `equilibrium` must generate two outputs: the output of the function and the Jacobian matrix. Table 1.1 illustrates the difference in computation times with and without gradient in the above example for certain parameter values. In most cases, the computation time is faster when the user provides the Jacobian matrix to the `fsolve` function.

T	Time (s) with gradient	Time (s) without gradient
50	0.047	0.061
100	0.088	0.159
200	0.34	0.51
500	1.93	2.79
1000	14.4	12.3

Table 1.1: Average computation times over 20 repetitions with and without gradient.

1.5.6 Gauss-Seidel Algorithm

The method described above to solve for an equilibrium can be computationally costly to implement. Assume, for example, that we have an economy with endogenous labor and different types of assets such that, for every period t, we have three or more optimality conditions. Assume also that the transition from the initial state k_0 to the steady state takes 30 or more periods. This means that in such an economy we would have to find the root of a system of 90+ equations. This can take quite some time, so the method described may not be very efficient.

In the Gauss-Seidel algorithm, instead of solving for all equilibrium conditions at the same time, we solve successively the system corresponding to every period t. In this way,

instead of solving the system of all equations at the same time, we solve many systems of equations, one after the other.

To implement the algorithm, we can simplify the system of equations Ψ_1 and Ψ_2 by solving for c_t in the budget constraint and plugging it in the Euler equation in the following way:

$$\Psi(k_t, k_{t+1}, k_{t+2}) = \frac{u'((1-\delta)k_t + F_l(k_t,1) + F_k(k_t,1)k_t - k_{t+1})}{\beta u'((1-\delta)k_{t+1} + F_l(k_{t+1},1) + F_k(k_{t+1},1)k_{t+1} - k_{t+2})} - (1-\delta+F_k(k_{t+1},1)) = 0$$

Now, to find an equilibrium we must find the capital sequence that satisfies the system of equations $\Psi(k_t, k_{t+1}, k_{t+2})$ for every t. The following algorithm does that:

Algorithm 2 (Gauss-Seidel)**.**

1. *Set $s = 0$, and start with a guess $k_1^0, k_2^0, \ldots, k_{T-1}^0$ for the sequence of capital along the transition, starting from k_0 and ending in $k_T = k^*$.*

2. *For every $j \in \{1, \ldots, T-1\}$, find the root x^* of $\Psi(k_{j-1}^s, x, k_{j+1}^s)$, such that $\Psi(k_{j-1}^s, x^*, k_{j+1}^s) = 0$. Set $k_j^{s+1} = x^*$.*

3. *If $\left\|(k_1^{s+1}, \ldots, k_{T-1}^{s+1}) - (k_1^s, \ldots, k_{T-1}^s)\right\| < \epsilon$, stop. Else, set $s = s+1$ and go to 2.*

Once we have found the sequence of capital along the transition from k_0 to k^*, we can construct the sequence of all other variables using the production function, the budget constraint, the capital evolution equation, and so on:

$$\begin{aligned}
y_t &= f(k_t, 1) \\
c_t &= (1-\delta)k_t + F_l(k_t,1) + F_k(k_t,1)k_t - k_{t+1} \\
i_t &= k_{+1} - (1-\delta)k_t \\
&\vdots
\end{aligned}$$

1.6 Recursive Representation and Dynamic Programming

The main problem with the model described in Sections 1.1-1.4 is that problems such as the household's, described by equation (1.17), require maximizing the utility function over an infinite space. The solution to the consumer's problem are infinite sequences $\{c_t, k_t, i_t\}_{t=0}^{\infty}$ for consumption, capital and investment. When the functional forms are such that we do not have closed-form solutions, as the one obtained for the log utility case in Section 1.1.4, the problem can become infeasible to estimate. The purpose of this section is to transform the consumer's problem in such a way that it can be solved with the use of dynamic programming.

1.6.1 Social Planner's Problem

First, let's take the social planner's problem (1.6) and define the function V in the following way:

$$V(k_0) \quad := \quad \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0 \text{ given}}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \tag{1.96}$$

$$c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1)$$

The function V is the maximum lifetime discounted utility attainable by an individual that starts life with capital level equal to k_0 and chooses optimally every period. Henceforth, we are going to call the function V the **value function**. Note that the maximum utility attainable V is a function of the stock of capital k_0 owned by the individual at the beginning of her life. In our setup, it can be shown that the value function is increasing in k_0; a consumer that starts life with a larger level of capital will be able to consume more over her life. We refer to k_0 as an **individual state variable**. In this problem, the only state variable is k_0, as this is the only value needed in period 0 to compute the lifetime utility and policy functions.

Now, let's rewrite the social planner's problem in the following way:

$$
\begin{aligned}
V(k_0) &= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \\
& \quad c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1) \\
&= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0 \text{ given}}} u(c_0) + \beta \cdot \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \quad s.t. \\
& \quad c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1) \\
&= \max_{\substack{c_0, k_1 \geq 0 \\ k_0 \text{ given}}} u(c_0) + \beta \cdot \left(\max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 1 \\ k_1 \text{ given}}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right) \quad s.t. \\
& \quad c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1) \\
&= \max_{c_0, k_1} u(c_0) + \beta \cdot \left(\max_{\substack{c_{t+1}, k_{t+2} \geq 0 \\ t \geq 0 \\ k_1 \text{ given}}} \sum_{t=0}^{\infty} \beta^{t} u(c_{t+1}) \right) \quad s.t. \\
& \quad c_t + k_{t+1} = (1-\delta)k_t + F(k_t, 1)
\end{aligned}
\tag{1.97}
$$

Note the similarities between the problem inside brackets in the last line and the problem in the first line of equation (1.97). Given that, in our environment, the utility and production functions remain constant over time, these two problems are equivalent to each other, except for the fact the in the first line the initial level of capital is given by k_0, while in the problem inside brackets the initial level of capital is given by k_1. This means that the problem inside brackets is simply $V(k_1)$, and we can rewrite the original problem as:

$$
\begin{aligned}
V(k_0) = \max_{c_0, k_1 \geq 0} \quad & u(c_0) + \beta V(k_1) \quad s.t. \\
& c_0 + k_1 = (1-\delta)k_0 + F(k_0, 1)
\end{aligned}
$$

To simplify notation, we will not use subscripts of time and, instead, will denote variables

in the present period without apostrophe $(c, k, \ldots)$, and variables that correspond to one period ahead with one apostrophe $(c', k', \ldots)$. In this way, the social planner's problem is:

$$V(k) = \max_{c,k' \geq 0} \qquad u(c) + \beta V(k') \quad s.t. \tag{1.98}$$

$$c + k' = (1 - \delta)k + F(k, 1)$$

This form of writing the social planner's problem is called the **recursive formulation** and can be shown to be equivalent to the original problem. The interpretation of this equation is that the maximum attainable lifetime utility is the sum of today's maximum utility $u(c)$ plus the discounted maximum utility attainable in the future $\beta V(k')$, given today's choice for k'. This problem is equivalent to the original problem, because it is equivalent to choose simultaneously the lifetime consumption and capital accumulation streams $\{c_t, k_{t+1}\}_{t=0}^{\infty}$, or to choose optimally today's consumption and capital accumulation, given that you are going to choose optimally in the future. Equation (1.98) is called the **Bellman equation**.

What are the unknowns in the Bellman equation? The Bellman equation is a *functional equation*, which means that, as opposed to the equations we are used to in which unknowns are scalars or vectors, in this case the unknowns are functions. There are three unknowns. The first unknown is a function $V(k)$, which gives the maximum utility for every level of individual capital k. This function appears on both sides of the equation, so solving for it means finding a function V that satisfies (1.98). Second, in order to achieve the utility level $V(k)$, we need to know what are the optimal levels of consumption and capital accumulation for every capital stock k. This means that the solution to the Bellman equation are three functions of k: $V(k)$, $c(k)$ and $k'(k)$. These last two functions are called **policy functions**, as they instruct the social planner what to choose, given a capital stock k.

Note that, given that the recursive representation is equivalent to the original problem (1.6), we can construct the value function, and lifetime consumption and capital accumu-

lation sequences using the policy functions:

$$
\begin{aligned}
c_0 &= c(k_0), & k_1 &= k'(k_0) \\
c_1 &= c(k_1), & k_2 &= k'(k_1) = k'(k'(k_0)) \\
c_2 &= c(k_2), & k_3 &= k'(k_2) = k'(k'(k_1)) = k'(k'(k'(k_0))) \\
& \vdots
\end{aligned}
$$

Also, we know that the total lifetime utility, given k_0, is given by:

$$V(k_0) = \sum_{t=0}^{\infty} \beta^t u(c_t)$$

The matter of how to solve the Bellman equation is the subject of Section 1.6.3. The techniques used to solve for it are a consequence of mathematical results on functional analysis, that ensure that the methods used yield a correct solution. A review of the mathematical results is the subject of another section.

1.6.2 Recursive Competitive Equilibrium

Having defined the recursive formulation of the social planner, we can define the equivalent concept for the household's problem (1.17). There are two main differences. First, in the competitive equilibrium the household has access to risk-free bonds a_t, which determine the lifetime consumption and utility, so they must be included as an individual state variable.

Second, as opposed to the social planner, who only needs to know the current level of capital k_0 of the household to determine lifetime discounted utility and policy functions, in a competitive equilibrium the household also needs to know what the prices in the economy will be for every t. Recall the prices are determined by aggregate capital in the economy, such that $w_t = F_l(K_t, 1), R_t = F_k(K_t, 1)$, and $r_t = R_t - \delta$. This means that the household needs to know what the aggregate level of capital will be in the economy, to be able to forecast correctly the sequence of prices. Therefore, the relevant state variables in the competitive equilibrium set-up are individual capital k_0, individual risk-free bonds a_0, and aggregate capital in the economy K_t every period.

We have to explicitly separate individual and aggregate capital because, since we assumed there is a continuum of households, individual capital k_t need not be exactly equal to

aggregate capital K_t. In equilibrium, in a representative agent economy, they will be equal to each other, but ex-ante they are not. Therefore, when describing the state variables we must distinguish between individual and aggregate state variables. In this case, the state variables are k_0, a_0, and K_t for every t.

We can write the household's problem as:

$$
\begin{aligned}
V(k_0, a_0, \{K_t\}_{t=0}^{\infty}) &= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0, a_0 \text{ given}}} \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t. \qquad (1.99) \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0 \\
&= \max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 0 \\ k_0, a_0 \text{ given}}} u(c_0) + \beta \cdot \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \quad s.t. \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0 \\
&= \max_{\substack{c_0, k_1 \geq 0 \\ k_0, a_0 \text{ given}}} u(c_0) + \beta \cdot \left(\max_{\substack{c_t, k_{t+1} \geq 0 \\ t \geq 1 \\ k_1, a_1 \text{ given}}} \sum_{t=1}^{\infty} \beta^{t-1} u(c_t) \right) \quad s.t. \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0 \\
&= \max_{c_0, k_1} u(c_0) + \beta \cdot \left(\max_{\substack{c_{t+1}, k_{t+2} \geq 0 \\ t \geq 0 \\ k_1, a_1 \text{ given}}} \sum_{t=0}^{\infty} \beta^{t} u(c_{t+1}) \right) \quad s.t. \\
& \quad c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1+r_t)a_t \\
& \quad \lim_{T\to\infty} \left(\prod_{t=0}^{T} \frac{1}{1+r_t} \right) a_{T+1} = 0
\end{aligned}
$$

The household's problem in recursive form is:

$$V(a,k,K) = \max_{c,k'\geq 0,a'} \quad u(c) + \beta V(a',k',K') \quad s.t. \tag{1.100}$$

$$c + k' + a' = (1-\delta)k + w(K) + R(K)k + (1+r(K))a$$

$$K' = H(K)$$

Again, we will denote k and a as individual state variables, and K as an **aggregate state variable**. Recall that in this economy there is a continuum of households, where each household realizes that it is small enough, so its decisions do not affect aggregates in the economy. For this reason, prices in the economy $w(K), r(K)$ depend on aggregate capital K, which is independent on the choice of capital of the household k.

The last equation in this recursive formulation is called an **aggregate law of motion**, and states the evolution of aggregate capital given by H. We assume that each household has perfect foresight and can perfectly forecast how aggregate capital will evolve -that is, every household knows H-, and knows that the evolution is independent of its own decisions. Solving for a recursive competitive equilibrium will entail solving for a function $H(K)$, as well as for $V(a,k,K), c(a,k,K), k'(a,k,K)$ and $a'(a,k,K)$.

We can now define a recursive competitive equilibrium in this setting:

Definition 3 (Recursive Competitive Equilibrium)**.** *A recursive competitive equilibrium are a value function $V(a,k,K)$, policy functions $c(a,k,K), k'(a,k,K), a'(a,k,K)$, pricing functions $w(K), R(K), r(K)$, and an aggregate law of motion $H(K)$ such that:*

1. *Given (a,k,K) and functions $w(K), R(K), r(K), H(K)$, the value function V solves the problem of the household (1.100), with $c(a,k,K), k'(a,k,K), a'(a,k,K)$ being the corresponding policy functions*

2. *Given K, prices are such that:*

$$\begin{aligned} w(K) &= F_L(K,L) \\ R(K) &= F_K(K,L) \\ r(K) &= R(K) - \delta \end{aligned}$$

3. *Given K, the prices $w(K), R(K), r(K)$ are such that the markets clear:*

$$c(A, K, K) + k'(A, K, K) = (1 - \delta)K + F(K, L)$$
$$a'(A, K, K) = 0$$

4. *For every K, the aggregate law of motion $H(K)$ is consistent with the optimal choice of households:*

$$H(K) = k'(A, K, K)$$

The first two points in this definition are simply the fact that households and firms maximize. The market clearing conditions are such that, in equilibrium, when individual state variables are equal to aggregate state variables ($k = K$ and $a = A$), markets clear. The last condition states that the law of motion that households use to forecast the evolution of capital coincides with the individual household's decisions, such that the forecasting is correct. Next section describes a method to numerically solve the problems described so far.

1.6.3 Value Function Iteration

In Section 1.1.4, we could solve analytically the model and find closed-form solutions for all the variables over time. However, this is uncommon as often, this type of models are much more convoluted and do not allow for a closed-form solution. In this section we explain the most common method to solve a Bellman equation, such as (1.98) and (1.100). This method always works under certain conditions, that will be explained later. The main idea of the value function iteration approach is to start with a guess of the value function for the household, and iterate until the problem converges. The algorithm is as follows:

Algorithm 3 (Value Function Iteration)**.**

1. *Start with a guess for the value function $V_0(k)$ for the social planner's problem (1.98). For example, set $V_0(k) = 0, \forall k \geq 0$.*

2. *Compute an updated guess $V_1(k)$, by solving the right-hand side of the Bellman equa-*

tion using $V_0(k)$:

$$V_1(k) = \max_{c,k' \geq 0} \qquad u(c) + \beta V_0(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

This can be computed directly, as now there is nothing unknown on the right-hand side. For the particular case where $V_0(k) = 0$, note that it suffices to find c and k' that maximize:

$$u((1-\delta)k + F(k,1) - k')$$

3. *In general, for $n \geq 1$, compute $V_{n+1}(k)$ using the guess $V_n(k)$ and the equation:*

$$V_{n+1}(k) = \max_{c,k' \geq 0} \qquad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

If $\|V_{n+1} - V_n\| < \epsilon$, stop and set the policy functions c and k' by:

$$\{c(k), k'(k)\} = \arg\max_{c,k' \geq 0} \qquad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

Otherwise, set $n = n + 1$ and repeat 3.

Under very general conditions, the algorithm described above works and the value function converges to a unique function V^*. This means that we will be able to solve the model and compute the value and policy functions, even though we don't have a closed form solution for them.

But, even if this algorithm seems simple, how can we perform this computation on a computer? It turns out that even with a computer it is not possible to find an exact solution, so the best we can aim for is to compute an approximation to the value function. In particular, we will only be able to solve the problem for a finite number of points k. The algorithm to approximate the solution is as follows:

1. Set a grid for the state variables $K = \{k_1, k_2, \ldots, k_m\}$ of size m.

2. Set $n = 0$ and start with a guess for the value function $V_0(k), k \in \{k_1, \ldots, k_m\}$.

3. For $n \geq 1$, compute $V_{n+1}(k), k \in \{k_1, \ldots, k_m\}$ using the Bellman equation:

$$V_{n+1}(k) = \max_{c,k' \geq 0} \quad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

   If $\sum_{i=1}^{m} |V_{n+1}(k_i) - V_n(k_i)| < \epsilon$, stop and set the policy functions c and k' by:

$$\{c(k), k'(k)\} = \arg\max_{c,k' \geq 0} \quad u(c) + \beta V_n(k') \quad s.t.$$

$$c + k' = (1-\delta)k + F(k,1)$$

   Otherwise, set $n = n + 1$ and repeat 3.

For example, let's take an economy where the utility function is logarithmic $u(c) = \ln(c)$ and the production function takes the Cobb-Douglas form, such that $F(k,1) = k^\alpha$. Assume that the parameters are $\alpha = 0.3$, $\beta = 0.95$, and $\delta = 1$. Finally, the grid for capital choices is given by the set $\mathcal{K} = \{0.04, 0.08, 0.12, 0.16, 0.2\}$. The purpose is to compute the value function at each of these capital grid points, such that we end up with an estimate of the values of $V(0.04), V(0.08), V(0.12), V(0.16)$, and $V(0.2)$. The algorithm goes as follows:

1. Start with a guess $V_0(k) = 0$ for all $k \in \mathcal{K}$.

2. Define $V_1(k)$ by solving:

$$\begin{aligned} V_1(k) &= \max_{k' \geq 0} u\left((1-\delta)k + F(k,1) - k'\right) + \beta V_0(k') \\ &= \max_{k' \geq 0} u\left(F(k,1) - k'\right) \end{aligned}$$

   Given that $V_0(k) = 0$ for all k, it is optimal to choose the minimum level of capital in

the grid, so $k'(k) = 0.04$ for all $k \in \mathcal{K}$. Therefore:

$$\begin{aligned}
V_1(0.04) &= u\left(0.04^{0.3} - 0.04\right) = -1.077 \\
V_1(0.08) &= u\left(0.08^{0.3} - 0.04\right) = -0.847 \\
V_1(0.12) &= u\left(0.12^{0.3} - 0.04\right) = -0.715 \\
V_1(0.16) &= u\left(0.16^{0.3} - 0.04\right) = -0.622 \\
V_1(0.20) &= u\left(0.20^{0.3} - 0.04\right) = -0.55
\end{aligned}$$

3. Define $V_2(k)$ by solving:

$$V_2(k) = \max_{k' \geq 0} u\left((1-\delta)k + F(k,1) - k'\right) + \beta V_1(k')$$

Now, given that $V_1(k) \neq 0$, for every $k \in \mathcal{K}$ we have to evaluate the utility for every $k' \in \mathcal{K}$. If the planner chooses $k' = 0.04$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.04\right) + \beta V_1(0.04)$$

If the planner chooses $k = 0.08$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.08\right) + \beta V_1(0.08)$$

If the planner chooses $k = 0.12$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.12\right) + \beta V_1(0.12)$$

If the planner chooses $k = 0.16$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.16\right) + \beta V_1(0.16)$$

If the planner chooses $k = 0.20$:

$$V_2(0.04) = u\left(0.04^{0.3} - 0.20\right) + \beta V_1(0.20)$$

The optimal choice is given by the highest of these. In particular, for $k = 0.04$ it is optimal to choose $k' = 0.08$ and $v_2(0.08) = -1.710$. This has to be done for each

$k \in \mathcal{K}$.

4. Continue until convergence. That is, until the point in which V_{n+1} is sufficiently close to V_n.

A natural question that arises is how to choose the grid for capital and how many points m to use. There is no correct answer to this. A larger number of grid points improves the accuracy obtained in the approximation of the value function V, but at the cost of a slower computation speed.

Note also that there are values for k that are not feasible. Take problem (1.98). The maximization on the right hand side of the equation is made over all k' that satisfy two constraints:

$$\begin{aligned} k' &\geq 0 \\ k' &\leq (1-\delta)k + F(k,1) \end{aligned}$$

The second inequality comes from the fact that $c \geq 0$. This means that choosing levels of capital outside the interval $[0, (1-\delta)k + F(k,1)]$ is not feasible, so we can restrict the points in the grid to being in that interval.

1.6.4 Dynamic Programming

This section is a review of the theorems that ensure that the value function iteration algorithm converges, and states some properties of the value and policy functions.

COMPLETAR!!!!

1.6.5 Problems

1. Write the recursive formulation of the social planner and household's problems when there is population growth

2. Write the recursive formulation of the social planner and household's problems when there is endogenous labor supply

3. Write the recursive formulation of the social planner and household's problems when there is exogenous technological growth

4. So far, we have assumed that markets are **sequential**, which means that every period there are markets for goods, factors of production, and savings. At every t, households receive wages w_t, returns to last period's capital $(1-\delta+R_t)k_t$, and returns to savings $(1+r_t)a_t$, and choose consumption c_t, capital k_{t+1}, and savings a_{t+1} for next period.

   There is an equivalent way of studying markets, which will prove useful later in the course. Assume agents start life at period $t=0$ and have perfect information about future income streams and prices in the economy. At $t=0$, before any consumption is realized or any income received, agents meet at the market and trade *all* future consumption and capital accumulation streams, subject to a *lifetime* budget constraint. Once trade has occurred, agents own perfectly enforceable contracts every period on consumption and capital agreed in period $t=0$.

   Let p_t denote the price, in terms of consumption in period $t=0$, of one unit of consumption to be delivered in period t (normalize $p_0=1$). Agents act competitively, which means that in period $t=0$ they take future prices $\{p_t\}_{t=0}^{\infty}$ as given and beyond their control when making decisions.

   The problem of the household is the following:

$$\max_{c_t,k_{t+1}} \quad \sum_{t=0}^{\infty}\beta^t u\left(c_t\right) \quad s.t.$$

$$\sum_{t=0}^{\infty} p_t(c_t+k_{t+1}) = \sum_{t=0}^{\infty} p_t((1-\delta)k_t + w_t + R_t k_t)$$

$$c_t, k_{t+1} \geq 0, \quad k_0 \text{ given}$$

The equilibrium in this environment is called an **Arrow-Debreu (A-D) equilibrium**.

(a) Solve for the A-D equilibrium

(b) What is the relationship between prices p_t and the interest rate on risk-free bonds r_t?

(c) Show the equivalence between the sequential markets and the A-D equilibria (*Hint:* use the no-Ponzi condition as stated in the class notes)

5. Let an economy where the period utility is given by $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, and the production function is $F(k,l) = k^{\alpha}l^{1-\alpha}$. Assume that individuals can only invest in capital (i.e. there are no risk-free bonds) and let the parameters take the following values:

$$\beta = 0.99; \quad \sigma = 2; \quad \delta = 0.05; \quad \alpha = 0.3;$$

(a) What is the value of steady state capital k^*, consumption c^*, investment i^*, production y^* and factor prices w^* and R^*?

(b) Set $T = 10$ and compute the transitional dynamics using one of the methods studied in class. Assuming that the initial condition is $k_0 = 0.1 \cdot k_{ss}$, plot consumption c_t, investment i_t, production y_t, wages w_t, rental rate R_t, and the ratio $\frac{i_t}{y_t}$ on the path to steady state

(c) How does the ratio $\frac{i_t}{y_t}$ compare to the Solow model?

6. Let an economy where the period utility is given by $u(c) = \log(c)$, the production function is $F(k,l) = k^{\alpha}l^{1-\alpha}$, and there is full depreciation of capital $\delta = 1$. Guess that the value function takes the form:

$$v(k) = A + B\log(k)$$

In this environment, the social planner's problem can be written as:

$$v(k) = \max_{k'} \quad \log(k^{\alpha} - k') + \beta v(k')$$

Solve the social planner's problem to find the values of A and B, and find the **policy function** $k'(k)$. How does this model compare to the Solow model?

7. Using the same functional forms as in question 2, assume that the economy is in steady state. Solve the social planner's problem using the **value function iteration** method. Use a grid for capital of 500 equally spaced points, from $0.5k_{ss}$ to $1.5k_{ss}$. Plot the value function, and the capital and consumption policy functions.

1.7 Growth Accounting

So far, we have assumed that output in the economy is produced with the use of two factors, labor L and capital K, which, combined with the available technology A, produce $Y_t = F(A_t, K_t, L_t)$ units of the final good. We would like to understand how does the growth of each of the factors of production and technology contribute to the growth of output in the economy. Is the economy growing due to an increase in the available capital? Is there technological growth? The exercise of decomposing output growth into the growth rates of technology, capital, and labor is called **growth accounting**, an was first proposed by Solow (1957).

The neoclassical production function satisfies the assumptions described in Section 1.1, which are mainly that the function $F(K, L)$ is increasing in both arguments ($F_K, F_L > 0$), concave ($F_{KK}, F_{LL} < 0$, $F_{KL} > 0$) and has constant returns to scale ($F(\lambda K, \lambda L) = \lambda F(K, L)$). With technological growth, the neoclassical production function can take the form of $F(AK, L)$ or $F(K, AL)$, depending on whether the technological growth augments capital or labor. If we assume that technical change is labor-augmenting and the production function is Cobb-Douglas, then $F(K, AL)$ can be rewritten as:

$$Y_t = F(K_t, A_t L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha} = A_t^{1-\alpha} K_t^{\alpha} L_t^{1-\alpha} = \tilde{A}_t F(K_t, L_t) \tag{1.101}$$

where $\tilde{A}_t = A_t^{1-\alpha}$. We denote $\hat{A}_t$ as the **total factor productivity** or TFP. Taking logarithms, we can rewrite equation (1.101) as:

$$\log(Y_t) = \log(\tilde{A}_t) + \alpha \log(K_t) + (1-\alpha)\log(L_t)$$

Note that α and $(1-\alpha)$ are the contributions of capital and labor to GDP. The variable $\tilde{A}_t$ captures everything else that affects production different from labor and physical capital. The term $\log(\tilde{A}_t)$ is commonly known as the **Solow residual**. This equation can be rewritten in per-worker terms, using the fact that $y_t = F(k_t, 1) = \tilde{A}_t k_t^{\alpha}$:

$$\log(y_t) = \log(\tilde{A}_t) + \alpha \log(k_t) \tag{1.102}$$

If we take equation (1.102) and subtract two consecutive periods:

$$\log\left(\frac{y_{t+1}}{y_t}\right) = \log\left(\frac{\tilde{A}_{t+1}}{\tilde{A}_t}\right) + \alpha \log\left(\frac{k_{t+1}}{k_t}\right) \tag{1.103}$$

Defining g_y, g_A, and g_k as the growth rates of output per worker, total factor productivity, and capital per worker, respectively, the following holds:

$$\log(1+g_y) = \log(1+g_A) + \alpha \log(1+g_k) \tag{1.104}$$

To compute the total factor productivity $\tilde{A}_t$ and the contribution of capital to output α, we need data on output Y_t, capital K_t, and labor L_t to use equation (1.103), where $k_t := K_t/L_t$ and $y_t := Y_t/L_t$. However, what does capital K_t stand for in the data? To construct a time series of the capital stock in the economy, we can use the **perpetual inventory method** which works as follows. Set an initial level of capital K_0 for the base year and assume a depreciation rate δ, which can be an estimate from the literature, usually between 0.02 and 0.05. At every period t, the total investment in the economy at constant prices I_t is reported in the national accounts, so we can construct the series of capital using the capital accumulation equation:

$$K_{t+1} = (1-\delta)K_t + I_t$$

By recursively plugging in this equation at every t, we have that:

$$K_t = (1-\delta)^t K_0 + \sum_{j=0}^{t-1}(1-\delta)^{t-1-j} I_j$$

As t increases, the level of K_0 becomes irrelevant, as it is multiplied by the term $(1-\delta)^t$, which tends to zero as t increases. Once the series for capital $\{K_t\}_{t=0}^T$ is constructed, we have all the data to compute α and $\{\tilde{A}_t\}_{t=0}^T$. First, to compute α, we can estimate the following regression:

$$\log\left(\frac{y_{t+1}}{y_t}\right) = \beta_0 + \alpha \log\left(\frac{k_{t+1}}{k_t}\right) + \epsilon_{t+1}$$

This will give us an estimate $\hat{\alpha}$ of the contribution of capital to GDP. A second approach to obtain an estimate for α is to note that, assuming that markets are competitive and that

production takes the Cobb-Douglas form:

$$r_t = F_K(K_t, L_t) = \alpha \tilde{A}_t K_t^{\alpha-1} L_t^{1-\alpha} = \alpha \tilde{A}_t K_t^{\alpha-1} L_t^{1-\alpha} \cdot \frac{K_t}{K_t} = \frac{\alpha \tilde{A}_t K_t^{\alpha} L_t^{1-\alpha}}{K_t} = \frac{\alpha Y_t}{K_t}$$

$$\Rightarrow \quad \alpha = \frac{r_t K_t}{Y_t} \tag{1.105}$$

Once we have an estimate for α, we can construct the time series for the TFP with:

$$\log(\tilde{A}_t) = \hat{\alpha} \log(k_t) - \log(y_t) \tag{1.106}$$

Using equation (1.104), the average growth of the TFP is given by:

$$g_A = e^{\log(1+g_y) - \hat{\alpha} \log(1+g_k)} - 1 \tag{1.107}$$

The contributions of TFP and capital per-worker growth to output growth are given by:

$$\text{Contribution of TFP growth} = \frac{\log(1+g_A)}{\log(1+g_y)}$$

$$\text{Contribution of capital growth} = \hat{\alpha} \cdot \frac{\log(1+g_A)}{\log(1+g_y)}$$

There are multiple problems with growth accounting. First, the measure of capital used aggregates different types of capital. However, this ignores that the quality of capital across different sectors may evolve at a different rate, and this evolution is not considered in our construction of the capital time series. Second, there are types of capital which are being ignored, such as human capital or managerial capital. Certainly, the production functions depend on the level of human capital of workers, on the quality of the managers, and so on. Ignoring these inputs represents a problem. Third, there are market imperfections that we are ruling out by assuming that markets are competitive. For instance, externalities and public goods represent a problem to growth accounting. Finally, the perpetual inventory method constructs the capital time series based on prices of capital over time, and not on quality. This poses a problem as, for example, prices of computers have declined significantly over the past years, while quality has increased. According to the perpetual inventory method, which uses prices to construct the capital stock, this would imply a smaller growth

in the stock of capital.

Chapter 2

Endogenous Growth Models

Sections 1.1-1.7 dealt mainly with the neoclassical growth model, where technological progress was assumed exogenous. Namely, we assumed that labor productivity A_t grew exogenously at a constant rate g, such that $A_{t+1} = (1+g)A_t$. Introducing this productivity growth allowed the model to generate growth when the economy reached the balanced growth path. That is, this model was able to explain long-run growth of economies. However, this model is not useful to understand where does productivity growth come from. Why is there productivity growth at all? What is the engine of productivity growth? Why does productivity grow faster in some nations than in others? The purpose of this section is to try to understand the source of productivity growth and what are its determinants. This will allow us to talk about policies that governments can implement to increase productivity growth in the long run.

The following subsections present different models in which there is long-run growth. The first three sections do not explicitly model the creation of new technologies or the expansion of the productivity A_t. Instead, these models remove some of the assumptions made so far, which prevent the economy from experiencing long-run growth. Namely, in the first two sections, we remove the assumption of decreasing returns to scale on capital. The third section studies a model in which there are externalities in capital accumulation. The last two sections present models in which the productivity growth is explicitly modelled.

2.1 The AK Model

Up to now we have assumed that the production function F had constant returns to scale, was strictly concave in both of its inputs K and L, and satisfied the Inada conditions. The most commonly used production function that satisfies these assumptions is the Cobb-Douglas function $F(K_t, L_t) = K_t^{\alpha}(A_t L_t)^{1-\alpha}$. In this section, we will assume that $\alpha = 1$, such that $F(K_t, L_t) = AK_t$, where A is a constant technological parameter, often normalized to 1. This function has constant returns on capital, as opposed to the diminishing returns to capital that the previous Cobb-Douglas specification satisfied. Also, production now does not depend on labor L_t, so wages are $w_t = 0$. In equilibrium, the capital rental rate will be exactly $R_t = A$ for every t.

Solving the social planner's problem yields the usual first order conditions:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1-\delta+A)}{(1+n)}$$

$$c_t + (1+n)k_{t+1} = (1-\delta)k_t + Ak_t$$

First, note that in this model there is generally no steady state. That is, there is no point in which variables per-capita remain constant. For instance, for consumption to be constant over time, we would need the following equality to hold:

$$\frac{1}{\beta} = \frac{(1-\delta+A)}{(1+n)}$$

This equation does not hold in general, unless the model parameters are specifically chosen to make it hold. Having dismissed the existence of a steady state, let's analyse the cases in which the economy reaches a balanced growth path. That is, a point in which all per-capita variables grow at a constant rate. Recall that the existence of a balanced growth path requires that the utility function satisfies the CRRA assumption, in which case the Euler equation becomes:

$$\frac{1}{\beta}\left(\frac{c_t}{c_{t+1}}\right)^{-\sigma} = \frac{(1-\delta+A)}{(1+n)} \tag{2.1}$$

This equation implies that the growth of consumption is always constant, such that $c_{t+1} =$

$(1+\gamma_c)c_t$. In a balanced growth path, γ_c is then pinned down by the following equation:

$$\frac{1}{\beta}(1+\gamma_c)^{\sigma} = \frac{(1-\delta+A)}{(1+n)} \tag{2.2}$$

That is, the growth rate of consumption along the balanced growth path is:

$$\gamma_c = \left(\beta \cdot \frac{(1-\delta+A)}{1+n}\right)^{\frac{1}{\sigma}} - 1 \tag{2.3}$$

This growth rate can be positive, even without technological growth. In particular, it is positive when β is sufficiently large, given that individuals are very patient and choose to save a large part of their income, so capital is rapidly accumulated over time and the constant returns to scale ensure that production continues to grow linearly – the marginal product of capital does not decrease over time. The same happens if depreciation δ is low, as capital accumulates at a fast rate. Finally, if the households have a large elasticity of substitution (if σ is close to 0), they will prefer to postpone consumption for the future by accumulating more capital, leading to larger growth rates.

If we take logarithms to (2.3) and express $\beta = 1/(1+\rho)$:

$$\log(1+\gamma_c) = \frac{1}{\sigma}\left(\log(1-\delta+A) - \log(1+n) - \log(1+\rho)\right)$$

If $A-\delta$, ρ and n are sufficiently small (say, below 10%), the growth rate can be approximated as:

$$\gamma_c \approx \frac{1}{\sigma}(A-\delta-n-\rho) \tag{2.4}$$

Note that, according to the Euler equation (2.1), the growth rate of consumption is *always* constant, independently of the initial point. This means that in the AK model there is no trasition to a balanced growth path; the economy is always on a balanced growth path. Furthermore, as opposed to the neoclassical growth model, where all variables grow at the rate of technological change g, in the AK model the growth rate of the economy depends on other parameters: A, δ, n, ρ.

2.2 Human Capital

Up to now, we have assumed that firms only use unskilled labor L_t and capital K_t as factors of production, where labor is understood as the amount of physical workers. The limitation of this is that the amount of labor is bounded by the population size and households cannot increase their labor supply beyond that. Moreover, this assumption does not allow for improvements in the quality of labor. Certainly, workers have become more productive over the last century, given the increase in schooling and accumulation of human capital. This section introduces a model with human capital, where households can invest in improving the quality of labor they supply.

Assume that the production function depends on physical capital K_t and human capital H_t, such that $Y_t = F(K_t, H_t)$, where F satisfies the same assumptions stated in Section 1.1. Here, we assume there is no technological growth. Households can accumulate human capital in a similar way as they choose to accumulate physical capital. Namely, human capital evolves according to a human-capital evolution equation:

$$H_{t+1} = (1 - \delta_h)H_t + I_t^h$$

where δ_h is the depreciation rate of human capital and I_t^h are total investments in human capital. As before, physical capital evolves according to the usual capital evolution equation, where we denote I_t as total investments and δ as the depreciation rate of physical capital. The aggregate resource constraint is thus:

$$C_t + I_t + I_t^h = (1 - \delta)K_t + (1 - \delta_h)H_t + F(K_t, H_t)$$

In this resource constraint we are implicitly assuming that human capital can be converted with a linear technology into the final good, so total undepreciated capital $(1-\delta_h)H_t$ can be consumed. The model can be re-expressed in per-capita terms, such that the resource constraint is:

$$c_t + (1 + n)k_{t+1} + (1 + n)h_{t+1} = (1 - \delta)k_t + (1 - \delta_h)h_t + F(k_t, h_t)$$

Where $h_t = H_t/L_t$ is the average human capital per worker in this economy. The social planner's problem is:

$$\max_{c_t, k_{t+1}, h_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t) \quad s.t.$$

$$\forall t \in \{0, 1, \ldots\}: \quad c_t + (1+n)k_{t+1} + (1+n)h_{t+1} = (1-\delta)k_t + (1-\delta_h)h_t + F(k_t, h_t)$$

$$c_t, k_{t+1}, h_{t+1} \geq 0, \quad h_0, k_0 \text{ given}$$

Setting the appropriate lagrangean and solving for the optimality conditions yields:

$$\begin{aligned}
[c_t]: &\quad \beta^t u'(c_t) - \lambda_t = 0 \\
[k_{t+1}]: &\quad -(1+n)\lambda_t + \lambda_{t+1}\left(1 - \delta + F_k(k_{t+1}, h_{t+1})\right) = 0 \\
[h_{t+1}]: &\quad -(1+n)\lambda_t + \lambda_{t+1}\left(1 - \delta_h + F_h(k_{t+1}, h_{t+1})\right) = 0 \\
[\lambda_t]: &\quad (1-\delta)k_t + (1-\delta_h)h_t + F(k_t, h_t) - c_t - (1+n)(k_{t+1} + h_{t+1}) = 0
\end{aligned}$$

The Euler equations are:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta + F_k(k_{t+1}, h_{t+1}))}{(1+n)}$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1 - \delta_h + F_h(k_{t+1}, h_{t+1}))}{(1+n)}$$

These optimality conditions imply that, in the optimum, $F_k(k_{t+1}, h_{t+1}) - \delta = F_h(k_{t+1}, h_{t+1}) - \delta_h$, so the social planner is indifferent between investing in human or physical capital. In particular, using the same reasoning as in (1.71), we can re-write this condition as:

$$F_k\left(\frac{k_{t+1}}{h_{t+1}}, 1\right) - \delta = F_h\left(\frac{k_{t+1}}{h_{t+1}}, 1\right) - \delta_h \tag{2.5}$$

In the optimum, the social planner will choose a physical-to-human capital ratio that satisfies (2.5). In particular, the physical-to-human capital ratio should remain constant at every period t, independently on whether the economy has reached a balanced growth path or not. In addition, the fact that the physical-to-human capital ratio is constant, implies that human and physical capital must grow at exactly the same rate. Denote by $\kappa = \frac{k_{t+1}}{h_{t+1}}$,

so the Euler equation is:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = \frac{(1-\delta+F_k(\kappa,1))}{(1+n)}$$

On a balanced growth path, denote by γ_c the growth rate of consumption. Recall that we must assume that the utility function takes the CRRA form, so γ_c is pinned down by:

$$1+\gamma_c = \left(\frac{(1-\delta+F_k(\kappa,1))}{(1+n)}\right)^{\frac{1}{\sigma}}$$

If we assume that F takes the Cobb-Douglas form $F(k,h) = k^{\alpha}h^{1-\alpha}$, and that physical and human capital depreciate at the same rate, such that $\delta = \delta_h$, the optimal physical-to-human capital ratio is:

$$\kappa = \frac{\alpha}{(1-\alpha)}$$

And the growth rate of the economy is:

$$\gamma_c = \left(\frac{\left(1-\delta+\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)}{(1+n)}\right)^{\frac{1}{\sigma}} - 1$$

As in the AK model, depending on the parameters of the model, the economy can reach a balanced growth path with positive growth (compare to equation (2.3)). The reason for this result is that in this model, as in the AK model, there are *constant returns to scale to factors of production that can be accumulated.* In previous sections this was not the case, as labor could not be accumulated beyond population growth and there were diminishing returns to scale on capital.

2.3 Externalities in Production

In this model, firms produce according to the usual production function $F(K_t, A_tL_t)$, where there is a labor-augmenting technology A_t. In this section, growth of the technology parameter A_t is driven endogenously within the model, as opposed to Section 1.4 where technological growth was exogenous. In particular, we assume that technology A_t equals the average capital used across firms: $A_t = \bar{K}_t$. This means that $Y_t = F(K_t, \bar{K}_tL_t)$. If we assume a Cobb-Douglas production function: $Y_t = K_t^{\alpha}(\bar{K}_tL_t)^{1-\alpha}$.

Recall that in the economy there is a continuum of identical firms of size 1. This means that every firm acts independently of each other, and the decision of each single firm does not affect aggregate outcomes in the economy. In particular, if a firm decides to increase its capital, the average capital used by firms in the economy stays unmodified at the same level. Average capital is affected only when a set of firms of positive measure change their capital. Therefore, every firm takes average capital in the economy as given and beyond its control.

Given these assumptions, the production function has constant returns to scale at the firm level. If a single firm decides to increase all factors of production by a proportion λ, production increases in that same proportion: $F(\lambda K_t, A_t(\lambda L_t)) = \lambda F(K_t, A_t L_t)$. This happens because the decision of a single firm does not affect aggregate capital, so increasing labor and capital by a factor λ leaves $\bar{K}_t$ unchanged.

However, note that from the social planner's perspective, if the factors of production of all firms increase by a factor λ, total production increases more than that, as average capital in the economy will also increase. That is, if the social planner increases capital used by firms by a factor λ, not only does K_t increases, but also $\bar{K}_t$. In the case of a Cobb-Douglas production function:

$$F(\lambda K_t, \lambda \bar{K}_t \lambda L_t) = (\lambda K_t)^{\alpha} (\lambda \bar{K}_t \lambda L_t)^{1-\alpha} = \lambda^{2-\alpha} K_t^{\alpha} L_t^{1-\alpha} > \lambda F(K_t, \bar{K}_t L_t)$$

This means that the social planner faces increasing returns to scale on its factors or production. This is driven by the fact that average capital in the economy increases the productivity of labor, so increasing capital has an additional indirect effect in the economy, beyond the additional production of that capital.

What drives the difference is that there is an externality driven by capital accumulation. When firms accumulate more capital, they do not internalize the fact that increasing average capital also increases the productivity of all other firms in the economy. Therefore, the level of capital chosen privately by each firm is below the socially optimal level. In contrast, the social planner knows that increasing capital has a direct effect, by increasing production, but also an indirect effect, by increasing productivity. Therefore, the social planner chooses the socially optimal level of capital. In this context, the welfare theorems do not hold, so the allocations that solve the social planner's problem and the competitive equilibrium are

not equal to each other.

2.3.1 Social Planner's Problem

The social planner knows that there is a continuum of firms of size 1 so that, if all firms choose capital K_t, the average capital is $\bar{K}_t = K_t$. Therefore, the production function becomes:

$$Y_t = F(K_t, \bar{K}_t L_t) = K_t^{\alpha}(\bar{K}_t L_t)^{1-\alpha} = K_t L_t^{1-\alpha}$$

Assume there is no population growth, such that $L_t = 1$ for all t. We can write total per-capita production as $y_t = k_t$. This is exactly the AK model, with $A = 1$ and population growth $n = 0$. This means that along a balanced growth path all variables grow at the rate:

$$\gamma_c = (\beta(2-\delta))^{\frac{1}{\sigma}} - 1$$

2.3.2 Competitive Equilibrium

In a competitive equilibrium, households solve the same problem as in Section 1.101. Recall that if we want to study balanced growth paths, we have to assume that $u = \frac{c^{1-\sigma}}{1-\sigma}$, so the household's optimality conditions are:

$$\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = 1 - \delta + R_{t+1}$$

$$c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + w_t + R_t k_t + (1 + r_t)a_t$$

The firm's problem is:

$$\max_{K_t, L_t} \quad K_t^{\alpha}(\bar{K}_t L_t)^{1-\alpha} - w_t L_t - R_t K_t$$

So prices of factors of production are:

$$\begin{aligned} R_t &= \alpha K_t^{\alpha-1}(\bar{K}_t L_t)^{1-\alpha} \\ w_t &= (1-\alpha)K_t^{\alpha}\bar{K}_t^{1-\alpha}L_t^{-\alpha} \end{aligned}$$

In equilibrium, $\bar{K}_t = K_t$ and $L_t = 1$, so equilibrium prices become:

$$\begin{aligned} R_t &= \alpha \\ w_t &= (1-\alpha)K_t \end{aligned}$$

Plugging these equilibrium prices in the household's Euler equation yields:

$$\frac{1}{\beta}\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = 1 - \delta + \alpha$$

So the growth rate is:

$$\gamma_c = (\beta(1-\delta+\alpha))^{\frac{1}{\sigma}} - 1$$

The growth rate on a competitive equilibrium is smaller than under the social planner's solution, given that $1 - \delta + \alpha < 2 - \delta$, as $\alpha < 1$. The reason is that, given that household's do not internalize that their own capital accumulation increases overall productivity in the economy, they under-accumulate capital. In contrast, the social planner internalizes this and choose the socially optimal level of capital.

In this model, there is long-run growth because, even though households face decreasing returns to capital accumulation, there are constant returns to capital accumulation in the economy. Therefore, capital can be accumulated at a constant rate, generating production growth at that same rate in the long run.

2.4 R&D Models - Romer (1990)

This section describes the first model in which productivity growth is explicitly modelled. In this setting, productivity A_t is a function of the number of different varieties of intermediate goods used in production.

Every period t, the economy is composed of $N_t + 1$ production sectors: a firm that produces the final good Y_t, and N_t firms that produce intermediate goods $X_1, \ldots, X_{N_t}$. The final good production firm uses as inputs each of the N_t intermediate goods and labor, to produce Y_t according to the production function: $Y_t = \left(\sum_{i=1}^{N_t} X_{it}^{\alpha}\right) L_t^{1-\alpha}$. The final good is produced in a perfectly competitive market and is used for consumption and investment. The problem of the final good firm is:

$$\max_{X_1,\ldots,X_{N_t},L_t} \quad \left(\sum_{i=1}^{N_t} X_{it}^{\alpha}\right) L_t^{1-\alpha} - \sum_{i=1}^{N_t} P_{it} X_{it} - w_t L_t \tag{2.6}$$

Where P_{it} is the price of the intermediate good X_i in period t. The optimality conditions for the final good firm are:

$$w_t = (1-\alpha)\left(\sum_{i=1}^{N_t} X_{it}^{\alpha}\right) L_t^{-\alpha} \tag{2.7}$$

$$P_{it} = \alpha X_{it}^{\alpha-1} L_t^{1-\alpha}, \quad i \in \{1, \ldots, N_t\} \tag{2.8}$$

Intermediate good firms operate on a monopolistic market. To produce good X_{it}, the intermediate producer uses the final good as an input for production and faces a constant marginal cost of production equal to 1. Every period the producer of X_{it} chooses the price of its good P_{it}, to solve:

$$\pi_{it} = \max_{P_{it}} \quad (P_{it} - 1) X_{it}(P_{it}) \tag{2.9}$$

Where $X_{it}(P_{it})$ is the demand function for the intermediate good X_{it}, taken as given by the monopolist. Namely, the demand function can be obtained from (2.8):

$$X_{it}(P_{it}) = \left(\frac{\alpha}{P_{it}}\right)^{\frac{1}{1-\alpha}} \tag{2.10}$$

So the intermediate good producer solves:

$$\pi_{it} = \max_{P_{it}} \quad (P_{it} - 1)\left(\frac{\alpha}{P_{it}}\right)^{\frac{1}{1-\alpha}} \tag{2.11}$$

Which yields the optimality condition for the intermediate good producer i:

$$P_{it} = \frac{1}{\alpha} \quad \Rightarrow \quad \pi_{it} = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}} \tag{2.12}$$

Households can engage in the creation of a new intermediate good sector j, beyond the existing ones $1, \ldots, N_t$. The creation of a new sector has a fixed cost equal to $\eta > 0$, which has to be paid at period t, and will allow the household to produce the new intermediate good X_j starting from period $t+1$. The owner of sector j has lifetime monopolistic power over that sector, which means that the returns of creating a new sector j are equal to $\sum_{s=t+1}^{\infty}\left(\prod_{l=t}^{s}\frac{1}{1+r_l}\right)\pi_{j,s}$. Given that the production of intermediate goods generates positive profits, households will be attracted to create new sectors as long as total profits of owning a new sector are larger than the cost of creating the sector. The problem of the household is:

$$\max_{c_t, a_{t+}, N_{t+1}} \quad \sum_{t=0}^{\infty}\beta^t u(c_t), \quad s.t. \tag{2.13}$$

$$\forall t \in \{0, 1, \ldots\}: \quad c_t + a_{t+1} = w_t + (1+r_t)a_t + \sum_{i=1}^{N_t}\pi_{it} - \eta(N_{t+1} - N_t)$$

$$c_t, N_{t+1} \geq 0, a_0 = 0$$

Given that profits of each intermediate good firm are $\pi_{it} = (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}$, the household's problem can be re-written as:

$$\max_{c_t, a_{t+}, N_{t+1}} \quad \sum_{t=0}^{\infty}\beta^t u(c_t), \quad s.t. \tag{2.14}$$

$$\forall t \in \{0, 1, \ldots\}: \quad c_t + a_{t+1} = w_t + (1+r_t)a_t + (1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}N_t - \eta(N_{t+1} - N_t)$$

$$c_t, N_{t+1} \geq 0, a_0 = 0$$

Note that in this environment households can move wealth across time through two different channels. They can invest in risk-free bonds a_{t+1}, or they can invest in the creation of new

intermediate goods X_{jt}, which generates profits every period in the future. Solving problem (2.14) yields the following two Euler equations for the household:

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + r_{t+1} \tag{2.15}$$

$$\frac{u'(c_t)}{\beta u'(c_{t+1})} = 1 + \frac{(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta} \tag{2.16}$$

These two equations imply a non-arbitrage condition between risk-free assets and the creation of new intermediate sectors, such that:

$$r_t = \frac{(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta} \tag{2.17}$$

That is, the returns to an additional risk-free bond a_{t+1} are equal, in equilibrium, to the returns of creating a new sector, net of the cost of creating that sector η. On a balanced growth path, where the interest rate is constant $r_t = r^*$, this non-arbitrage condition is equivalent to a zero-profit condition on the creation of a new intermediate good sector, given by the free entry condition:

$$\underbrace{\eta}_{\text{Cost of creating new sector}} = \underbrace{\sum_{s=t+1}^{\infty} \left(\prod_{l=t}^{s} \frac{1}{1+r^*} \right) \pi_{jt}}_{\text{Benefit of creating new sector}} \tag{2.18}$$

That is, in equilibrium the household is indifferent between accumulating an additional risk-free bond, or creating a new intermediate good sector. Along a balanced growth path, the growth rate of consumption is given by:

$$\gamma_c = \left(\beta \left(1 + \frac{(1-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta} \right) \right)^{1/\sigma} - 1 \tag{2.19}$$

Note that in this economy the welfare theorems do not hold, given that intermediate good firms have monopolistic power. This means that intermediate goods are produced below the socially-optimal level. The social planner produces intermediate goods up to the socially optimal level of production, achieving larger growth rates in the long run.

Chapter 3

Models with Uncertainty

The models we have studied so far have no uncertainty at all. However, many of the interesting questions in Economics arise because of the existence of uncertainty. People save as a precaution in case of receiving negative income shocks in the future, households choose the amount of risk they want to bear to optimally choose their portfolio, the decision to buy a house or rent depends partly on the behavior of house prices, and so on. Incorporating uncertainty allows economists to model a wide variety of economic problems.

Although in real life there are many different sources of uncertainty, in our models we abstract from different sources and study some particular channels. The most common sources of uncertainty in economic models are productivity shocks, that directly affect the technology of production, shocks to household's preferences, which affect the tastes for different goods, or shocks to policy, such as shocks to government expenditures.

3.1 Representation of Risk

In this section we formalize the concept of risk and the information space of agents. Assume there is a stochastic variable z in the economy. That is, households have uncertainty regarding the realization of z every period in the future. This stochastic variable may represent different variables. For example, z can represent the realization of the firm's productivity, such that: $F(z, k, l) = e^z k^\alpha l^{1-\alpha}$; the realization of a taste shock, such that $u(c_t) = zc_t^{1-\sigma}/1 - \sigma$; or the realization of government spending, such that $g_t = z$. Assume that $z \in Z$, such that Z is the set of all possible events or outcomes that can occur. Although easily generalizable, we will assume henceforth that the stochastic variable z is discrete and,

without loss of generality, the set $Z = \{1, 2, \ldots, n\}$ is finite.

We denote a history of shocks up to period t as $z^t = (z_0, z_1, \ldots, z_t)$, where the superscript represents a history, or a vector of shocks, and the subscript represents a realization of the shock in that period. We also denote the set of all possible histories of shocks up to period t as $Z^t = Z \times Z \times \ldots \times Z$. The probability of observing history z^t in period t is $\pi(z^t)$, such that:

$$0 \leq \pi(z^t) \leq 1, \quad \sum_{z^t \in Z^t} \pi(z^t) = 1$$

In the models presented here, we assume that households observe current and past realizations of the shocks, and know the probability distribution of shocks. This means that in period t the household knows the realized $z^t \in Z^t$, and the probabilities $\pi(z^j)$ for all $j \geq t$. The tree in Figure 3.1 illustrates the possible realizations of histories over time, where $Z = \{1, 2\}$ and there are two periods.

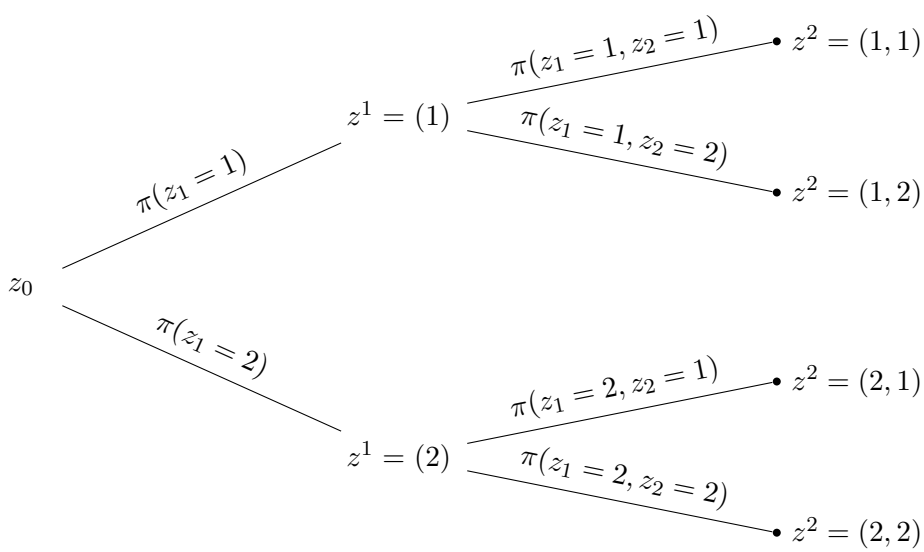


Figure 3.1: Tree of possible histories $z^t \in Z^t$.

Note that the number of possible histories grows exponentially over time. If $|Z|$ denotes the number of possible outcomes in the set Z, the number of possible histories up to period t is equal to $|Z|^t$. This makes the models with uncertainty computationally much more complex than the models in past sections.

Every period, households take their decisions after having observed the realization of the

shock history z^t. This means that, every period, all endogenous variables in the economy can potentially depend on z^t. For example, the optimal consumption chosen by the household will probably be low if the firm's productivity realizations up to period t have been $z^t = \{1, 1, \ldots, 1\}$ – that is, if productivity takes the lowest possible values every period up to t. In contrast, consumption will probably be higher if the realizations of the shocks up to t are $z^t = \{n, n, \ldots, n\}$. Therefore, when describing the household and firm's problem, we must explicitly denote endogenous variables as functions of z^t. For example, after observing history z^t, the household will choose consumption $c_t(z^t)$. In general, a random variable x that depends on the shock history z^t will be denoted as $x_t(z^t)$. Given that the probability distribution of shocks is known by agents in the economy, the expected value of x in period t can be computed as:

$$\mathbb{E}_0 x_t(z^t) = \sum_{z^t \in Z^t} \pi(z^t) x_t(z^t)$$

Similarly, we can compute the expected utility at period t as:

$$\mathbb{E}_0 u(c_t(z^t)) = \sum_{z^t \in Z^t} \pi(z^t) u(c_t(z^t))$$

Finally, we denote by $Z^t | z^{t-s}$ the set of all histories up to period t that follow the history z^{t-s}. Using Bayes' rule, the conditional expectation of $x_t(z^t)$, given that history z^{t-s} has been realized, is:

$$\mathbb{E}_{t-s}(x_t(z^t)|z^{t-s}) = \sum_{z^t \in Z^t | z^{t-s}} \frac{\pi(z^t)}{\pi(z^{t-s})} x(z^t)$$

3.2 Social Planner's Problem

In this section, assume that the stochastic variable $z \in Z$ represents productivity shocks, such that $Y_t = e^{z_t} F(K_t, L_t)$, where F satisfies the same conditions as in past sections. For simplicity we assume there is no population growth, so $L_t(z^t) = 1$, for all $z^t \in Z^t$ and $t \geq 0$, and $Y_t = e^{z_t} F(K_t, 1)$.

Every period t, the social planner (and agents in the economy) observes the realization of the productivity shock z_t, as well as the history of past shocks z^t, but does not know

what the future realizations will be. The planner only knows the stochastic process from which the realizations are drawn, and assigns probabilities $\pi(z^t)$ to each possible history in the future.

The social planner chooses optimal consumption and investment decisions that are contingent on the realization of the shock, to maximize the expected lifetime utility:

$$\mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t(z^t)) = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \pi(z^t) \beta^t u(c_t(z^t))$$

As already described, optimal choices for consumption, savings, and capital are functions of the shock history z^t, not only of the present realization of the shock z_t. For example, the choices of the planner are going to be different after five consecutive periods of negative productivity shocks, than after just one period of low productivity. In this sense, what matters are shock histories, rather than the single realization of the shock in a period.

The social planner's problem is a natural extension of problem (1.6):

$$\max_{\substack{c_t(z^t), k_{t+1}(z^t) \\ t \geq 0 \quad z^t \in Z^t}} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(c_t(z^t)) = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) \quad s.t. \tag{3.1}$$

$$c_t(z^t) + k_{t+1}(z^t) = (1-\delta) k_t(z^{t-1}) + e^{z_t} F(k_t(z^{t-1}), 1)$$

$$c_t(z^t), k_{t+1}(z^t) \geq 0, \quad k_0 \text{ given}$$

Various things are worth noting. First, the planner chooses at period t and history z^t, the capital to take to $t+1$, so $k_{t+1}(z^t)$ is a function of z^t. For this reason, the capital k_t that the individual owns in period t was chosen the period before, so $k_t(z^{t-1})$ is a function of z^{t-1}, which is the history that precedes z^t. Second, given that capital is not state-contingent, $k_{t+1}(z^t)$ appears at $t+1$ in *every* single history that follows after z^t, so the first-order condition for $[k_{t+1}(z^t)]$ will sum over all possible histories $z^{t+1}|z^t$. What happens is that now the individual chooses a capital amount $k_{t+1}(z^t)$ in period t, without knowing what the realization of z^{t+1} is going to be. Therefore, the first order condition sums over all possible realizations of z in period $t+1$. Third, the choices of c_t and k_{t+1} depend on the whole history z^t and not only on the last realization of the shock z_t. Finally, if the size of the set of possible realizations Z is large, the number of possible histories grows exponentially with time. In this sense, computing such a model might be infeasible. Later sections describe the

computation in this context. The lagrangean associated with the social planner's problem is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) + \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \lambda(z^t) \quad \big[(1-\delta)k_t(z^{t-1}) + e^{z_t} F(k_t(z^{t-1}), 1) + \\ -c_t(z^t) - k_{t+1}(z^t)\big]$$

The social planner optimizes with respect to consumption and capital accumulation *at every possible state of the world*. That is, the first order conditions must be taken with respect to c and k at every possible z^t. In equilibrium, for every $z^t \in Z^t$ and every $t \geq 0$:

$$[c_t(z^t)]: \qquad \beta^t \pi(z^t) u'(c_t(z^t)) - \lambda(z^t) = 0$$

$$[k_{t+1}(z^t)]: \qquad -\lambda(z^t) + \sum_{z^{t+1} \in Z^{t+1}|z^t} \lambda(z^{t+1}) \left(1 - \delta + e^{z_{t+1}} F_k(k_{t+1}(z^t), 1)\right) = 0$$

$$[\lambda(z^t)]: \qquad (1-\delta)k_t(z^{t-1}) + e^{z_t} F(k_t(z^{t-1}), 1) - c_t(z^t) - k_{t+1}(z^t) = 0$$

Which yields the following optimality conditions, for every $z^t \in Z^t$ and every $t \geq 0$:

Euler eq: $$u'(c_t(z^t)) = \beta \sum_{z^{t+1} \in Z^{t+1}|z^t} \left(1 - \delta + e^{z_{t+1}} F_k(k_{t+1}(z^t), 1)\right) \frac{\pi(z^{t+1})}{\pi(z^t)} u'(c_{t+1}(z^{t+1}))$$

Resource cons: $$c_t(z^t) + k_{t+1}(z^t) = (1-\delta)k_t(z^{t-1}) + e^{z_t} F(k_t(z^{t-1}), 1)$$

Transv. cond.: $$\lim_{t \to \infty} \quad \beta^t \sum_{z^t \in Z^t} \pi(z^t) u'(c_t(z^t)) k_{t+1}(z^t) = 0$$

Init. cond.: k_0 given

The Euler equation can be re-written as:

$$u'(c_t(z^t)) = \beta \mathbb{E}_t \left(1 - \delta + e^{z_t} F_k(k_t(z^{t-1}), 1) u'(c_{t+1}(z^{t+1})) | z^t\right)$$

Where $\mathbb{E}_t$ is the expectations operator that includes all the information up to period t. This Euler equation relates consumption in period t, with *expected* consumption in period $t+1$. This is a consequence of the fact that the social planner cannot transfer wealth from z^t to a specific state z^{t+1}; if the planner chooses capital $k_{t+1}(z^t)$, no matter what z_{t+1} is, the same amount of capital will be available at $t+1$.

Note that in this environment we cannot talk about a steady state, as variables are expected to move stochastically every period. That is, there is no "long-run" notion in which the economy is expected to settle and variables to converge, given the nature of the stochastic process. In the example above, it is expected that, even in the long-run, the stochastic productivity will follow the same stochastic process.

Next section describes a competitive equilibrium in this environment. As in the neoclassical growth model without uncertainty, given the lack of frictions, the allocations of the social planner's problem are the same as those of the competitive equilibrium, which are the subject of next section.

3.3 Competitive Equilibrium: Incomplete Markets

As in Section 1.1, we assume that households have access to two assets to transfer wealth from t to $t+1$: physical capital k and risk-free bonds a. In this section, we assume that these assets are not state-dependent, which means that the amount of assets owned does not depend on the realization of the shock z. This will become clear once we formulate the household's problem. The problem of the household is a natural extension of the problem without uncertainty:

$$\max_{\substack{c_t(z^t), a_{t+1}(z^t), k_{t+1}(z^t) \\ t\geq 0 \quad z^t \in Z^t}} \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) \quad s.t. \tag{3.2}$$

$$\begin{aligned}
& c_t(z^t) + a_{t+1}(z^t) + k_{t+1}(z^t) = \\
& \qquad (1-\delta)k_t(z^{t-1}) + w_t(z^t) + R_t(z^t)k_t(z^{t-1}) + (1+r_t(z^t))a_t(z^{t-1}) \\
& a_{t+1}(z^t) \geq -\bar{A} \\
& c_t(z^t), k_{t+1}(z^t) \geq 0, \quad k_0, a_0 = 0 \text{ given}
\end{aligned}$$

Here we are assuming that the individual chooses capital $k_{t+1}(z^t)$ and risk-free assets

$a_{t+1}(z^t)$ at period t and history z^t, to take to period $t+1$ irrespective of the realization of the shock at $t+1$. That is, no matter what the realization z_{t+1} is at $t+1$, the household will wake up with $k_{t+1}(z^t)$ and $a_{t+1}(z^t)$. This means that the household cannot insure against the shocks and the returns to its investment are not state-contingent. We refer to this as an **incomplete markets** set-up, as there are no insurance markets for every state of the world. Later, we will discuss how a market in which the individual can insure against different realizations of the shock would look like.

In the case where the shock affects firm's productivity, why are the choices of the household dependent on the realization of the shock? Even though the budget constraint is not directly affected by z_t, there are general equilibrium effects that affect household choices. As can be expected, the wages w and returns to investments r, in equilibrium, will be pinned down by:

$$w_t(z^t) = e^{z_t} F_l(k_t(z^t), l_t(z^t))$$
$$R_t(z^t) = e^{z_t} F_k(k_t(z^t), l_t(z^t))$$

This means that a low productivity shock will imply low wages and low returns to investment, which will affect household's decisions. Something similar will happen when we have shocks that affect directly the household's decisions, such as a taste shock. In that case, even though the firm might not be directly affected by the shock, in equilibrium it will.

The firm's problem can also be naturally extended to the case with uncertainty:

$$\begin{aligned} \pi_t(z^t) &= \max_{k_t(z^t), l_t(z^t)} \quad y_t(z^t) - w_t(z^t) l_t(z^t) - r_t(z^t) k_t(z^t) \\ &= \max_{k_t(z^t), l_t(z^t)} \quad e^{z_t} F(k_t(z^t), l_t(z^t)) - w_t(z^t) l_t(z^t) - R_t(z^t) k_t(z^t) \end{aligned} \tag{3.3}$$

Definition 4 (Competitive Equilibrium)**.** *A competitive equilibrium are allocations for the household* $\{c_t(z^t), k^s_{t+1}(z^t), a_{t+1}(z^t)\}_{t=0, z^t \in Z^t}^{\infty}$*, allocations for the firm* $\{k^d_t(z^t), l_t(z^t)\}_{t=0, z^t \in Z^t}^{\infty}$*, and prices* $\{w_t(z^t), R_t(z^t), r_t(z^t)\}_{t=0, z^t \in Z^t}^{\infty}$ *such that:*

1. *Given* k_0 *and prices* $\{w_t(z^t), R_t(z^t), r_t(z^t)\}_{t=0, z^t \in Z^t}^{\infty}$*, the allocations* $\{c_t(z^t), k^s_{t+1}(z^t), a_{t+1}(z^t)\}_{t=0, z^t \in Z^t}^{\infty}$ *solve the optimization problem of the household, described by (3.2).*

2. *At every* $t \in \{0, 1, \ldots\}$ *and* $z^t \in Z^t$*, given prices* $w_t(z^t), R_t(z^t)$*, the allocations* $k_t^d(z^t), l_t(z^t)$ *solve the optimization problem of the firm, described by (3.3)*

3. *Markets clear for all* $t \geq 0$ *and* $z^t \in Z^t$*:*

   (a) *Goods:* $c_t(z^t) + k_{t+1}^s(z^t) = (1-\delta)k_t^s(z^{t-1}) + F(k_t^d(z^t), l_t(z^t))$

   (b) *Labor:* $l_t(z^t) = 1$

   (c) *Capital:* $k_t^d(z^t) = k_t^s(z^{t-1})$

   (d) *Risk-free bonds:* $a_t(z^{t-1}) = 0$

To solve for the equilibrium, let's solve first the household's problem. The associated lagrangean is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) + \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \lambda(z^t) \big[(1-\delta)k_t(z^{t-1}) + w_t(z^t) + R_t(z^t)k_t(z^{t-1}) + (1 + r_t(z^t))a_t(z^{t-1}) - c_t(z^t) - k_{t+1}(z^t) - a_{t+1}(z^t)\big]$$

The first order conditions are:

$$[c_t(z^t)]: \quad \beta^t \pi(z^t) u'(c_t(z^t)) - \lambda(z^t) = 0$$

$$[k_{t+1}(z^t)]: \quad -\lambda(z^t) + \sum_{z^{t+1} \in Z^{t+1}|z^t} \lambda(z^{t+1}) \big(1 - \delta + R_{t+1}(z^{t+1})\big) = 0$$

$$[a_{t+1}(z^t)]: \quad -\lambda(z^t) + \sum_{z^{t+1} \in Z^{t+1}|z^t} \lambda(z^{t+1}) \big(1 + r_{t+1}(z^{t+1})\big) = 0$$

$$[\lambda(z^t)]: \quad (1-\delta)k_t(z^{t-1}) + w_t(z^t) + r_t(z^t)k_t(z^{t-1}) + \pi_t(z^t) - c_t(z^t) - k_{t+1}(z^t) = 0$$

These conditions are analogous to the first order conditions of problem (1.17) of the household without uncertainty, except for one thing: as in the social planner's problem, given that capital and risk-free bonds are not state-contingent, $k_{t+1}(z^t)$ and $a_{t+1}(z^t)$ appear at $t+1$ in *every* single history that follows after z^t, so the conditions $[k_{t+1}(z^t)]$ and $[a_{t+1}(z^t)]$ sum over all possible histories $z^{t+1}|z^t$. Combining these conditions, the Euler equations

become:

$$u'(c_t(z^t)) = \beta \sum_{z^{t+1} \in Z^{t+1}|z^t} \left(1 - \delta + R_{t+1}(z^{t+1})\right) \frac{\pi(z^{t+1})}{\pi(z^t)} u'(c_{t+1}(z^{t+1}))$$

$$u'(c_t(z^t)) = \beta \sum_{z^{t+1} \in Z^{t+1}|z^t} \left(1 + r_{t+1}(z^{t+1})\right) \frac{\pi(z^{t+1})}{\pi(z^t)} u'(c_{t+1}(z^{t+1}))$$

Which, in terms of expectations become:

$$u'(c_t(z^t)) = \beta \mathbb{E}_t \left[\left(1 - \delta + R_{t+1}(z^{t+1})\right) u'(c_{t+1}(z^{t+1}))|z^t\right] \tag{3.4}$$

$$u'(c_t(z^t)) = \beta \mathbb{E}_t \left[\left(1 + r_{t+1}(z^{t+1})\right) u'(c_{t+1}(z^{t+1}))|z^t\right] \tag{3.5}$$

Note that now, as opposed to the case without uncertainty, there is no non-arbitrage condition that equates the returns to capital and assets. On average, returns to both assets are such that the marginal utilities at t and $t+1$ are related though these equations, but returns of both assets need not be the same at every state of the world.

Given a sequence of prices $\{w_t(z^t), r_t(z^t)\}_{t=0}^{\infty}$, the allocations $\{c_t(z^t), k_{t+1}(z^t)\}_{t=0}^{\infty}$ that satisfy the following equations characterize the optimal solution for the household:

Euler eq:

$$u'(c_t(z^t)) = \beta \sum_{z^{t+1} \in Z^{t+1}|z^t} \left(1 - \delta + R_{t+1}(z^{t+1})\right) \frac{\pi(z^{t+1})}{\pi(z^t)} u'(c_{t+1}(z^{t+1}))$$

$$u'(c_t(z^t)) = \beta \sum_{z^{t+1} \in Z^{t+1}|z^t} \left(1 + r_{t+1}(z^{t+1})\right) \frac{\pi(z^{t+1})}{\pi(z^t)} u'(c_{t+1}(z^{t+1}))$$

Bud. const:

$$c_t(z^t) + a_{t+1}(z^t) + k_{t+1}(z^t) = (1-\delta)k_t(z^{t-1}) + w_t(z^t) + R_t(z^t)k_t(z^{t-1}) + (1 + r_t(z^t))a_t(z^{t-1})$$

Transv. cond.:

$$\lim_{t\to\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u'(c_t(z^t)) k_{t+1}(z^t) = 0$$

$$\lim_{t\to\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u'(c_t(z^t)) a_{t+1}(z^t) = 0$$

Init. cond.: $k_0, a_0 = 0$ given

Using the optimality conditions of the households and firms, it can be shown that the solution to the social planner's problem and the competitive equilibrium are equivalent. Next section describes an environment with complete markets.

3.4 Competitive Equilibrium: Complete Markets

In last section, we assumed that households could only move wealth across time through a risk-free bond a, or capital k that were not state-contingent. That is, the household could not buy different units of bonds or capital for different states of the world in the future.

However, financial markets are usually richer and individuals can buy assets that pay only in certain states of the world. For example, health insurance is a state-contingent contract. If the individual gets sick, the insurer will pay any medical expenses. If the individual does not get sick, the insurer does not pay anything. This means that insurance is a financial asset bought in period t, that pays a different amount in every state of the world in $t+1$.

There are more complex financial instruments, such as options and derivatives, that are also state-contingent. For example, a call option gives the owner the opportunity to buy an asset at a predetermined price when the market price of the asset is above a specific threshold. This means that, under certain states of the world, namely when the price of the underlying financial asset is high, the owner of the option will be able to buy at a cheaper price and will receive the difference between the market price and the predetermined price in the option. That is, an option pays a positive payoff in states of the world where the price of the underlying asset is large, and zero otherwise.

A market in which individuals can buy perfectly state-contingent assets, or in which there exists an asset for every state of the world, is a **complete market**. In real life, markets are not perfectly complete, but developed financial markets such as the U.S. are closer to being a complete market.

An environment with complete asset markets is not very different than the markets we have analyzed so far. The only difference is that now agents have access to state-contingent assets. For exposition, let's denote the state-contingent assets as b. If at z^t the individual buys $b_{t+1}(z_{t+1})$ units of the bond in period t at a price per unit $q(z^t, z_{t+1})$, where z^{t+1} is a history that follows z^t, the individual receives $b_{t+1}(z_{t+1})$ if the history z^{t+1} is realized, and

zero otherwise. The state-contingent financial assets b are often called **Arrow-securities**. The problem of the household becomes:

$$\max_{\substack{c_t(z^t), k_{t+1}(z^t), \{b_{t+1}(z^{t+1})\}_{z_{t+1}\in Z} \\ t\geq 0 \quad z^t \in Z^t}} \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) \quad s.t. \tag{3.6}$$

$$c_t(z^t) + k_{t+1}(z^t) + \sum_{z_{t+1}\in Z} q(z^t, z_{t+1}) \cdot b_{t+1}(z^t, z_{t+1}) =$$
$$(1-\delta)k_t(z^{t-1}) + b_t(z^t) + w_t(z^t) + R_t(z^t)k_t(z^{t-1})$$

$$c_t(z^t), k_{t+1}(z^t) \geq 0, \quad b_{t+1}(z^{t+1}) \geq -B, \quad k_0, b_0 = 0 \text{ given}$$

In this problem, the individual buys state-contingent assets for every possible state of the world tomorrow, so the total expenditures include the sum of the expenditures on $q(z^t, z_{t+1}) \cdot b_{t+1}(z^t, z_{t+1})$ for every possible state z_{t+1} at $t+1$. Also, on the state of the world z^t, the household receives the amount $b_t(z^t)$, which is the number of bonds bought in period $t-1$ that pay only if z^t is realized.

The associated lagrangean is:

$$\mathcal{L} = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) + \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \lambda(z^t) \Bigg[(1-\delta)k_t(z^{t-1}) + b_t(z^t) + w_t(z^t) + R_t(z^t)k_t(z^{t-1})$$
$$- c_t(z^t) - k_{t+1}(z^t) - \sum_{z_{t+1}\in Z} q(z^t, z_{t+1}) \cdot b_{t+1}(z^t, z_{t+1}) \Bigg]$$

The first order conditions are:

$$
\begin{aligned}
[c_t(z^t)]: &\quad \beta^t \pi(z^t) u'(c_t(z^t)) - \lambda(z^t) = 0 \\
[k_{t+1}(z^t)]: &\quad -\lambda(z^t) + \sum_{z^{t+1} \in Z^{t+1}|z^t} \lambda(z^{t+1}) \left(1 - \delta + R_{t+1}(z^{t+1})\right) = 0 \\
[b_{t+1}(z^{t+1})]: &\quad -\lambda(z^t) q(z^t, z_{t+1}) + \lambda(z^{t+1}) = 0 \\
[\lambda(z^t)]: &\quad (1-\delta) k_t(z^{t-1}) + b_t(z^t) + w_t(z^t) + R_t(z^t) k_t(z^{t-1}) \\
&\qquad - c_t(z^t) - k_{t+1}(z^t) - \sum_{z_{t+1} \in Z} q(z^t, z_{t+1}) \cdot b_{t+1}(z^t, z_{t+1}) = 0
\end{aligned}
$$

With the presence of state-contingent assets, we have an additional optimality condition that pins down the price for the state-contingent assets, or Arrow securities:

$$
u'(c_t(z^t)) = \beta \cdot \frac{\pi(z^{t+1})}{\pi(z^t)} \frac{1}{q(z^t, z_{t+1})} \cdot u'(c_{t+1}(z^{t+1})), \qquad \forall z^{t+1} \in Z^{t+1}|z^t, \quad \forall z^t \in Z^t \tag{3.7}
$$

Note that this is nothing different than an Euler equation that relates consumption at history z^t in period t, with consumption at every posible history z^{t+1} in period $t+1$. As opposed to the Euler equation with incomplete markets, given by equations (3.4) and (3.5), in which marginal utility at t is related to *expected* marginal utility at $t+1$, with complete markets there is an Euler equation for every state in $t+1$ so there is no expectation operator $\mathbb{E}$. Given that the individual is able to perfectly insure against every state of the world at $t+1$, the household can perfectly smooth consumption over states of the world. The household's problem is completely characterized by the following equations for all $t \in \{0, 1, \ldots\}$, and for

all $z^t \in Z^t$:

$$u'(c_t(z^t)) = \beta \sum_{z^{t+1} \in Z^{t+1}|z^t} \left(1 - \delta + R_{t+1}(z^{t+1})\right) \frac{\pi(z^{t+1})}{\pi(z^t)} u'(c_{t+1}(z^{t+1})) \tag{3.8}$$

$$u'(c_t(z^t)) = \beta \cdot \frac{\pi(z^{t+1})}{\pi(z^t)} \cdot \frac{1}{q(z^t, z_{t+1})} u'(c_{t+1}(z^{t+1})) \tag{3.9}$$

$$\begin{aligned} c_t(z^t) + k_{t+1}(z^t) + \sum_{z_{t+1} \in Z} q(z^t, z_{t+1}) \cdot b_{t+1}(z^t, z_{t+1}) = \\ (1-\delta)k_t(z^{t-1}) + b_t(z^t) + w_t(z^t) + R_t(z^t)k_t(z^{t-1}) \end{aligned} \tag{3.10}$$

Moreover, given that capital is still non state-contingent, whereas Arrow securities are, there is no non-arbitrage condition linking the returns to capital to the returns of assets. By combining (3.8) and (3.9), the most we can get is:

$$1 = \sum_{z^{t+1} \in Z^{t+1}|z^t} \left(\frac{1 - \delta + R_{t+1}(z^{t+1})}{q(z^t, z_{t+1})} \right) \tag{3.11}$$

The definition of an equilibrium is analogous to the equilibrium in the incomplete markets case, with the only difference that now there is a market clearing condition for each of the Arrow securities, given that each is an independent asset. Again, since we are assuming that there is a representative agent, in equilibrium there is zero net-supply of each of the Arrow securities.

3.5 Arrow-Debreu Markets

Recall the environment defined in Problem 4 in Section 1.6. Without uncertainty, we were able to show that the Arrow-Debreu and the sequential formulations were equivalent to each other. In this section, we will define the Arrow-Debreu environment. It turns out that the Arrow-Debreu environment with uncertainty is equivalent to the sequential markets environment *with* complete markets. If households have access to Arrow securities, or state-contingent bonds, the equilibrium allocations will be equivalent to an Arrow Debreu equilibrium.

Assume agents have perfect information about the prices $w(z^t), R(z^t)$ at every state of the world z^t for every $t \geq 0$. That is, agents have perfect foresight about their income

streams in any state of the world. At $t = 0$, before any consumption is realized or any income received, agents meet at the market and trade *all* future consumption and capital accumulation streams, at every state of the world, subject to a *lifetime* budget constraint. Once trade occurs in period $t = 0$, agents own perfectly enforceable contracts every period to receive/deliver the consumption and capital quantities agreed upon in period $t = 0$. Denote the price of the consumption good delivered in state z^t, in terms of period 0 consumption, as $p(z^t)$. We normalize $p_0 = 1$. Agents act competitively, so when trading occurs they take prices $\{p(z^t)\}_{z^t \in Z^t, t \geq o}$ as given and beyond their control. The problem of the household is:

$$\max_{\substack{c_t(z^t), k_{t+1}(z^t) \\ t \geq 0 \quad z^t \in Z^t}} \quad \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} \beta^t \pi(z^t) u(c_t(z^t)) \quad s.t. \tag{3.12}$$

$$\sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p(z^t)(c_t(z^t) + k_{t+1}(z^t)) = \sum_{t=0}^{\infty} \sum_{z^t \in Z^t} p(z^t)((1-\delta)k_t(z^{t-1}) + w_t(z^t) + R_t(z^t)k_t(z^{t-1}))$$

$$c_t(z^t), k_{t+1}(z^t) \geq 0, \quad k_0 \text{ given}$$

The optimality conditions yield the following Euler equation:

$$u(c_t(z^t)) = \beta \frac{\pi(z^{t+1})}{\pi(z^t)} \frac{p(z^t)}{p(z^{t+1})} u'(c_{t+1}(z^{t+1})) \tag{3.13}$$

By normalizing the price at $t = 0$ to 1, we can derive an expression for prices:

$$p(z^t) = \beta^t \pi(z^t) \frac{u'(c_t(z^t))}{u'(c_0(z_0))} \tag{3.14}$$

Note that the allocations that solve the Arrow-Debreu and the sequential formulation with complete markets are equivalent, and the prices of Arrow q securities are related to p according to:

$$q(z^t, z_{t+1}) = \frac{p(z^{t+1})}{p(z^t)} \tag{3.15}$$

On the Arrow-Debreu environment, households can buy consumption and capital for *specific* states of the world. This means that they can insure by acquiring different bundles for different realizations if the shock z. This is equivalent to the complete markets environment, where households can transfer wealth across time and states of the world.

3.6 Markov Processes

In Section 1.6, we expressed the sequential household's problem in recursive form, which simplified the computation of the model. Now that we added uncertainty, it is useful to study the conditions under which the model can be represented recursively. Note that, even if the set of possible shocks Z is small, the set of possible histories grows exponentially with time. This means that if we were to compute the model in its sequential form, we would have to solve for an infeasible number of states.

To achieve this, we usually assume that the shock process follows a Markov process of order 1. This means that the conditional probability of a shock in $t+1$ only depends on the last realization of the shock, in period t, and not on the whole history up to that point:

$$P(z_{t+1}|z^t) = P(z_{t+1}|z_t)$$

This means that the value of the shock realization does not depend on the whole history, but only on the last realization. For example, let's assume that there are two possible technology shocks: $z_l = 0$ and $z_h = 1$. Every period t, the firm draws the corresponding shock z_t from $Z = (z_l, z_h)$. The probabilities of drawing each shock are given by:

$$P(z_{t+1} = 0|z_t = 0) = 0.7 \qquad P(z_{t+1} = 0|z_t = 1) = 0.1$$
$$P(z_{t+1} = 1|z_t = 0) = 0.3 \qquad P(z_{t+1} = 1|z_t = 1) = 0.9$$

In this example, the probabilities of drawing a high shock z_h or a low shock z_l only depend on the last value of the shock, and not on the shock history. For instance, at $t = 3$, the probability of drawing $z_3 = 1$ after the history $(z_0 = 1, z_1 = 0, z_2 = 1)$ is $P(z_3 = 1|z_2 = 1) = 0.9$, which is exactly the same as the probability of drawing $z_3 = 1$ after the histories $(z_0 = 0, z_1 = 0, z_2 = 1)$, $(z_0 = 1, z_1 = 1, z_2 = 1)$, and $(z_0 = 0, z_1 = 1, z_2 = 1)$.

An example of a continuous Markov process of order 1 are autoregressive processes AR(1):

$$z_{t+1} = \mu + \rho z_t + \epsilon_{t+1}, \qquad \epsilon_{t+1} \sim N(0, \sigma^2)$$

where ϵ_{t+1} is a random variable that is distributed normal with mean 0 and variance σ^2.

In this process, note that the value of the random variable z_{t+1} depends only on the immediately previous value z_t, and not on the history of shocks $(z_0, z_1, \ldots, z_t)$.

By assuming that shocks follow a Markov process of order 1, we ensure that the consumption, capital and investment decisions in our model depend only on the last draw of the shock, and not on the whole history. This will allow us to represent the sequential problems of the household and the social planner in a recursive form.

In general, let's assume that there are n possible values for the shock, such that, for every t, $z_t \in Z = (1, 2, \ldots, n)$. The **transition probability matrix** represents the transition probabilities from one state to another:

$$\Pi = \begin{bmatrix} P(z_{t+1} = 1|z_t = 1) & P(z_{t+1} = 2|z_t = 1) & \ldots & P(z_{t+1} = n|z_t = 1) \\ P(z_{t+1} = 1|z_t = 2) & P(z_{t+1} = 2|z_t = 2) & \ldots & P(z_{t+1} = n|z_t = 2) \\ \vdots & \vdots & \ddots & \vdots \\ P(z_{t+1} = 1|z_t = n) & P(z_{t+1} = 2|z_t = n) & \ldots & P(z_{t+1} = n|z_t = n) \end{bmatrix}$$

In the matrix Π, the index of the row represents the value of the last shock z_t, whereas the index of the column represent the value of the future realization of the shock z_{t+1}. In other words, the entry (i, j) of the matrix Π represents the probability that the shock is $z_{t+1} = j$ given that $z_t = i$. In our previous example, we would have:

$$\Pi = \begin{bmatrix} 0.7 & 0.3 \\ 0.1 & 0.9 \end{bmatrix} \tag{3.16}$$

Note that in every transition probability matrix, the sum of the values of each row i must be exactly equal to 1, as this is the sum of the probabilities that z_{t+1} takes on any value $z_{t+1} \in Z$, given that $z_t = i$. That is, for every $i \in \{1, \ldots, n\}$:

$$\sum_{j=1}^{n} \Pi_{i,j} = \sum_{j=1}^{n} P(z_{t+1} = j|z_t = i) = 1$$

Now, let's assume that the unconditional distribution of shocks at time t is given by the column vector $P_t = (p_t^1, \ldots, p_t^n)'$. That is, at a given period t, the probability of drawing the

probability shock $z_t = j$ is equal to p_t^j. How does the unconditional probability distribution P_t evolves to P_{t+1}? Given the transition probability matrix:

$$p_{t+1}^j = \sum_{k=1}^{n} \Pi_{k,j} \cdot p_t^k$$

This means that:

$$P_{t+1} = \Pi' \cdot P_t \qquad \Longleftrightarrow \qquad P'_{t+1} = P'_t \cdot \Pi$$

A **stationary or invariant distribution** $\bar{\Pi}$ for the Markov chain described by the transition probability matrix Π is an unconditional distribution such that:

$$\bar{\Pi} = \Pi' \cdot \bar{\Pi}$$

For example, let's take the transition probability matrix (3.16). The invariant distribution is $\bar{\Pi} = (p_1, p_2)'$ such that:

$$\bar{\Pi} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.7 & 0.1 \\ 0.3 & 0.9 \end{bmatrix} \cdot \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0.7p_1 + 0.1p_2 \\ 0.3p_1 + 0.9p_2 \end{bmatrix}$$

This, plus the fact that $p_1 + p_2 = 1$ yields $p_1 = 0.25, p_2 = 0.75$, so the invariant or stationary distribution is:

$$\bar{\Pi} = \begin{bmatrix} 0.25 \\ 0.75 \end{bmatrix}$$

A Markov process described by the transition probability matrix Π is stationary if, starting with an unconditional probability distribution P_0:

$$\lim_{t \to \infty} P_t = \bar{\Pi}$$

where $\bar{\Pi}$ is the invariant distribution associated with Π. It can be shown that if all the entries of the transition probability matrix Π are positive, there exists an unique invariant distribution and the unconditional distribution converges to it.

This means that, although we cannot speak about a steady state, given that even in the long run there will still be shock realizations, we know that in the long run the economy will converge to a point in which the shocks will follow the invariant distribution and unconditional probabilities will remain constant.

With the assumption that shocks follow a Markov process of order 1, we can now proceed to formulate the definition of a competitive equilibrium in recursive form, in the same fashion as Definition 3 in Section 1.6.2.

3.7 Recursive Representation

Having assumed that the shock follows a Markov process of order 1, we can now proceed to rewrite the social planner and household's problems in recursive terms. First, let's start with the social planner's problem with incomplete markets, defined by equation (3.1). In this case, there are two relevant state variables for the social planner: the level of capital in the economy $k_t(z_{t-1})$, and the present productivity shock z_t. Note that the productivity shock in $t+1$ is entirely determined by the shock at t, given that z follows a Markov process. Therefore, we do not need to include the entire history of shocks as a state variable. This problem can be recursively written in the following way:

$$V(k,z) = \max_{c,k' \geq 0} \quad u(c) + \beta \mathbb{E}_{z'|z} V(k',z') \quad s.t. \tag{3.17}$$

$$c + k' = (1-\delta)k + e^z F(k,1)$$

Once the shock z is realized, the function V that solves this problem is the maximum value that the social planner can attain, given z and k. This value function takes into account that productivity evolution is a stochastic process that evolves according to the distribution π, so the maximization problem chooses c, k' that maximize the present utility plus the *expected* value that can be attained in the next period. The expectation is taken over the possible values that the shock z' can attain next period: $\mathbb{E}_{z'|z} V(k',z') = \sum_{z' \in Z} \pi(z'|z) V(k',z')$. The transition probabilities are given by $\pi(z'|z)$, which represents the probability that the shock z' is realized tomorrow, given that the shock z was realized today.

Again, the solution to the social planner's problem are a value function $V(k,z)$ and policy functions $c(k,z)$ and $k'(k,z)$ that solve the Bellman equation. Using Blackwell's

conditions, it can be shown that the social planner's problem has a unique solution, which can be obtained with value function iteration.

In an analogous way, we can write the household's problem in a recursive way. Recall that, in contrast to the social planner, household's take into account prices to make their decisions. Moreover, prices depend on the aggregate labor and capital quantities in the economy, which are independent of, and cannot be influenced by, the individual consumers. Therefore, in the household's problem we have to distinguish between the aggregate state variables, which are aggregate capital K and the productivity shock that affects the whole economy z, and the individual state variable which is the level of capital k. The problem of the household is:

$$
\begin{aligned}
V(k,K,A,z) = \max_{c,k'\geq 0} \quad & u(c) + \beta \sum_{z'} \pi(z'|z) V(k',K',A',z') \quad s.t. \\
& c + k' + a' = (1-\delta)k + w(K,A,z) + r(K,A,z)k + (1+r(K,A,z))a \\
& K' = H_K(K,A,z) \\
& A' = H_A(K,A,z)
\end{aligned}
\tag{3.18}
$$

The function H represents the aggregate law of motion of capital in the economy and is known by the household. The definition of a stochastic recursive competitive equilibrium is the following:

Definition 5 (Stochastic Recursive Competitive Equilibrium)**.** *A stochastic recursive competitive equilibrium are a value function $V(k,K,z)$, policy functions $c(k,K,z), k'(k,K,z)$, pricing functions $w(K,z), r(K,z)$, and an aggregate law of motion $H(K,z)$, such that:*

1. *Given $w(K,z), r(K,z), H(K,z)$, the value function $V(k,K,z)$ solves the problem of the household (3.18), with $c(k,K,z), k'(k,K,z)$ being the corresponding policy functions.*

2. *Given (k,K,z), prices are such that:*

$$
\begin{aligned}
w(K,z) &= e^z F_l(K,1) \\
r(K,z) &= e^z F_k(K,1)
\end{aligned}
$$

3. *Given (K, z), the prices $w(K, z), r(K, z)$ are such that the markets clear:*

$$c(K, K, z) + i(K, K, z) = F(K, 1)$$

4. *For every (K, z), the aggregate law of motion $H(K, z)$ is consistent with the optimal choice of capital accumulation of households:*

$$K(K, z) = k'(K, K, z)$$

As before, we can recover the sequence of variables at every state of the world, by using the policy functions.

3.8 Solution Methods

This section describes the main solution methods to compute the model described so far. The value function iteration method solves the model globally, and the algorithm is a generalization of the method presented in Section 1.6.3. Then, I present two additional methods: log-linearization around steady state and perturbation.

3.8.1 Value Function Iteration

The value function iteration method with uncertainty is analogous to the method presented in Section 1.6.3 with more than one state variable. The only difference is that now the second state variable z evolves stochastically and is not chosen by the social planner/household. However, the assumption that z follows a Markov process with a transition probability matrix Π, makes the computation not very different from before.

As before, we have to define a grid for capital $\mathcal{K} = \{k_1, k_2, \ldots, k_m\}$ and a grid for shocks $\mathcal{Z} = \{z_1, \ldots, z_s\}$. The algorithm will approximate the value and policy functions at every point in the grids $\mathcal{K}, \mathcal{Z}$.

Algorithm 4 (Value Function Iteration)**.**

1. *Define a grid for capital $\mathcal{K} = \{k_1, k_2, \ldots, k_m\}$ and a grid for shocks $\mathcal{Z} = \{z_1, \ldots, z_s\}$.*

2. *Set $n = 0$ and start with a guess for the value function, such as $V_0(k, z) = 0$ for all $k \in \mathcal{K}, z \in \mathcal{Z}$.*

3. *For $n \geq 1$, compute an updated guess $V_{n+1}(k,z)$ by solving the right-hand side of the Bellman equation using $V_n(k,z)$:*

$$V_{n+1}(k,z) = \max_{c,k' \geq 0} \quad u(c) + \beta \sum_{z' \in \mathcal{Z}} \pi(z'|z) V_n(k',z') \quad s.t.$$

$$c + k' = (1-\delta)k + e^z F(k,1)$$

If $\|V_{n+1} - V_n\| = \sqrt{\sum_{k \in \mathcal{K}, z \in \mathcal{Z}} (V_{n+1}(k,z) - V_n(k,z))^2} < \epsilon$, stop and set the policy functions as:

$$\{c(k,z), k'(k,z)\} = \arg\max_{c,k' \geq 0} \quad u(c) + \beta \sum_{z' \in \mathcal{Z}} \pi(z'|z) V_n(k',z') \quad s.t.$$

$$c + k' = (1-\delta)k + e^z F(k,1)$$

Otherwise, set $n = n+1$ and repeat 3.

Technically speaking, the algorithm can be implemented in the following way. Construct a matrix V_{guess} of size $m \times s$, which are the dimensions of the grids for capital and productivity, such that:

$$V_{guess} = \begin{bmatrix} V_0(k_1,z_1) & V_0(k_1,z_2) & \dots & V_0(k_1,z_s) \\ V_0(k_2,z_1) & V_0(k_2,z_2) & \dots & V_0(k_2,z_s) \\ \vdots & & \ddots & \vdots \\ V_0(k_m,z_1) & V_0(k_m,z_2) & \dots & V_0(k_m,z_s) \end{bmatrix}$$

For every k and z, construct the vector $A(k,z)$ as:

$$A(k,z) = \begin{bmatrix} u((1-\delta)k + e^z F(k,1) - k_1) \\ u((1-\delta)k + e^z F(k,1) - k_2) \\ \vdots \\ u((1-\delta)k + e^z F(k,1) - k_m) \end{bmatrix} + \begin{bmatrix} V_{guess}(k_1,z_1) & \dots & V_{guess}(k_1,z_s) \\ V_{guess}(k_2,z_1) & \dots & V_{guess}(k_2,z_s) \\ \vdots & \ddots & \vdots \\ V_{guess}(k_m,z_1) & \dots & V_{guess}(k_m,z_s) \end{bmatrix} \cdot \begin{bmatrix} \pi(z_1|z) \\ \pi(z_2|z) \\ \vdots \\ \pi(z_s|z) \end{bmatrix}$$

Note that $A(k, z)$ is a vector of size m, which is the size of the grid for capital $\mathcal{K}$. Now, for every $k_i \in \{k_1, \ldots, k_m\}$, check whether $(1-\delta)k + e^z F(k, 1) - k_i \geq 0$. If not, replace the i-th entry of $A(k, z)$ with 10^{-6}, or a very large negative number. This is made to prevent choosing levels of capital k_i that imply a negative consumption, which is not feasible.

Finally, set $V_{update}(k, z) = \max A(k, z)$. That is, $V_{update}(k, z)$ should be assigned the value of the largest element in the vector $A(k, z)$. Repeat this for every (k, z) in the grids. If the euclidean distance between V_{guess} and V_{update} is small, the algorithm has converged. If, instead, the difference is above the tolerance level ϵ, set $V_{guess} = V_{update}$ and compute again the matrix V_{update} by computing $A(k, s), \ldots$

Once the algorithm converges, the policy function $k'(k, z)$ can be constructed for every (k, z) as the index of the vector $A(k, z)$ that has the largest entry.

3.8.2 Discretization of Stochastic Variables

So far, we have assumed that the stochastic process in our model is discrete and has finite support $\mathcal{Z}$, in order to implement the value function iteration algorithm. However, this might be a restrictive assumption, as we might be interested in computing models with AR(1) processes to describe the evolution of certain variables, such as productivity. This section reviews the main methods used in the literature to discretize continouos autorregressive processes, by Tauchen (1986) and Rouwenhorst (1995). The idea of these methods is to take an autorregressive process of order 1, which can be univariate or multivariate, and convert it into a discrete Markov process of degree 1. The purpose is to obtain a grid $\mathcal{Z}$ for possible values that the stochastic variable can take upon and a transition probability matrix Π.

These methods are of utmost importance, given that in economics we often assume that shocks certain shocks follow autorregressive processes. For example, following Storesletten et al. (2004), the literature often assumes that idiosyncratic labor productivity z_t follows an AR(1) process:

$$z_{t+1} = (1-\rho)\mu_z + \rho z_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, \sigma_\epsilon^2) \tag{3.19}$$

To numerically solve the social planner's problem, we should assume that z can take values on a finite grid, and evolves according to a transition probability matrix.

Tauchen (1986)'s Method - Univariate Process

Recall that $\mathbb{E}(z) = \mu_z$ and $Var(z) = \sigma_z^2 = \frac{\sigma_\epsilon^2}{(1-\rho^2)}$. The grid is chosen as a set of equally spaced points around the mean for the process μ_z. We must choose the width of the grid, or a parameter m, that defines the number of standard deviations around the mean that the grid will cover. That is, let l be the number of points in the grid, and define:

$$z_1 = \mu_z - m \cdot \sigma_z, \quad z_l = \mu_z + m \cdot \sigma_z$$

Define the distance between any two consecutive grid points as d, namely:

$$d := \frac{2m\sigma_z}{l}$$

This means that $z_2 = z_1 + d, z_3 = z_2 + d, \ldots$, such that the grid $\{z_1, \ldots, z_l\}$ is made up of equally spaced points. The transition probabilities from state z_i to state z_j are constructed as follows:

$$\begin{aligned}
\pi(z' = z_j | z = z_i) &= \Pr(z_j - d/2 < (1-\rho)\mu_z + \rho z_i + \epsilon' < z_j + d/2) \\
&= \Pr(z_j - d/2 - (1-\rho)\mu_z - \rho z_i < \epsilon' < z_j + d/2 - (1-\rho)\mu_z - \rho z_i)
\end{aligned}$$

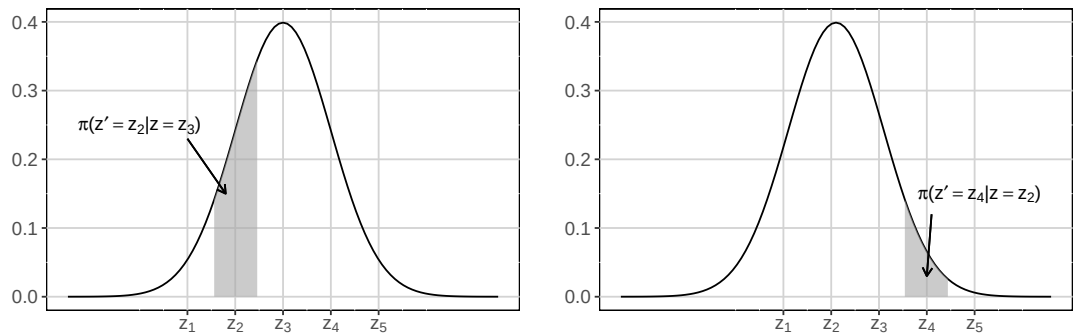
Figure 3.2 illustrates the transition probabilities from z_i to z_j, which are the area under the normal distribution surrounding point z_j, when the distribution is centered at ρz_i.

Then, for every $1 < j < l$, set:

$$\pi(z' = z_j | z = z_i) = \Theta\left(\frac{z_j + d/2 - (1-\rho)\mu_z - \rho z_i}{\sigma_\epsilon}\right) - \Theta\left(\frac{z_j - d/2 - (1-\rho)\mu_z - \rho z_i}{\sigma_\epsilon}\right)$$

Where Θ is the CDF of a standard normal distribution. If $j = 1$ or $j = l$, set:

$$\begin{aligned}
\pi(z' = z_1 | z = z_i) &= \Theta\left(\frac{z_j + d/2 - (1-\rho)\mu_z - \rho z_i}{\sigma_\epsilon}\right) \\
\pi(z' = z_l | z = z_i) &= 1 - \Theta\left(\frac{z_j - d/2 - (1-\rho)\mu_z - \rho z_i}{\sigma_\epsilon}\right)
\end{aligned}$$



(a) Transition probability from z_3 to z_2.

(b) Transition probability from z_2 to z_4.

Figure 3.2: Tauchen's transition probabilities.

As the number of grid points l increases, the approximation of the original process improves. With Tauchen's method, we can compute models with continuous AR(1) processes by first discretizing the process into a finite-state Markov process, and the applying the value function iteration method.

Tauchen (1986)'s Method - Multivariate Process

Assume now that the stochastic variables $z \in \mathbb{R}^K$ in our model evolve according to the VAR(1) process:

$$z' = Az + \epsilon', \quad \epsilon' \sim N(0, \Sigma_\epsilon) \tag{3.20}$$

Pick the number of grid points in each of the dimensions of the stochastic variable $n_1, \ldots, n_K$, so the state space of the stochastic variables has $n_1 \times n_2 \times \ldots \times n_K$ points. For each dimension $k \in \{1, \ldots, K\}$, define the grid points:

$$z_1^k = -m \cdot \sigma_{z_k}, \quad z_{n_k}^k = m \cdot \sigma_{z_k} \tag{3.21}$$

where $\sigma_{z_k}^2$ can be obtained from $vec(\Sigma_z) = (I - A \bigotimes A) vec(\Sigma_\epsilon)$. As with the univariate case, the remaining grid points are chosen to be equally spaced, such that $z_2^k = z_1^k + d_k, z_3^k = z_2^k + d_k, \ldots$, where $d_k = \frac{2m\sigma_{z_k}}{n_k}$. To compute the transition probability matrix, consider the transition from $(z_{i_1}^1, z_{i_2}^2, \ldots, z_{i_K}^K)$ to $(z_{j_1}^1, z_{j_2}^2, \ldots, z_{j_K}^K)$. For each dimension $k \in \{1, \ldots, K\}$, the probability of z^k transitioning from state i_k to state j_k is given by:

$$\pi^k_{i_k,j_k} = \begin{cases} \Phi\left(\frac{z^k_1 + d_k/2 - A_{kk} z^k_{i_k}}{\sigma_{\epsilon_k}}\right), & j_k = 1 \\ \Phi\left(\frac{z^k_{j_k} + d_k/2 - A_{kk} z^k_{i_k}}{\sigma_{\epsilon_k}}\right) - \Phi\left(\frac{z^k_{j_k} - d_k/2 - A_{kk} z^k_{i_k}}{\sigma_{\epsilon_k}}\right), & 1 < j_k < N_k \\ 1 - \Phi\left(\frac{z^k_{n_k} - d_k/2 - A_{kk} z^k_{i_k}}{\sigma_{\epsilon_k}}\right), & j_k = N_k \end{cases} \tag{3.22}$$

So the probability of the transition from $z_i = (z^1_{i_1}, z^2_{i_2}, \ldots, z^K_{i_K})$ to $z_j = (z^1_{j_1}, z^2_{j_2}, \ldots, z^K_{j_K})$ is equal to $\pi(z' = z_i | z = z_j) = \prod_{k=1}^K \pi^k_{i_k,j_k}$

Rouwenhorst (1995)'s Method - Univariate Process

Consider the AR(1) process for z with $\mu_z = 0$:

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0, \sigma^2_\epsilon)$$

Note that you can always re-express an arbitrary AR(1) process of the form (3.19) with $\mu_z \neq 0$, as a process with zero mean, by re-defining the stochastic variable in terms of deviations from the mean. Denote by l the number of grid points, and set the endpoints of the grid as:

$$z_1 = -\sigma_z \cdot \sqrt{l-1}, \quad z_l = \sigma_z \cdot \sqrt{l-1}$$

Where, again, the other points on the grid are set equidistantly, such that $z_2 = z_1 + d, z_3 = z_2 + d, \ldots$, and $d = \frac{2\sigma_z \cdot \sqrt{l-1}}{l}$. The transition probability matrix is defined recursively in the following way:

Algorithm 5.

1. *Define $p = q = \frac{1+\rho}{2}$, under the assumption of a symmetric distribution, and:*

$$\pi_2 := \begin{pmatrix} p & 1-p \\ 1-q & q \end{pmatrix}$$

2. *For $n \geq 2$, define:*

$$\pi_{n+1} := p \begin{pmatrix} \pi_n & \mathbf{0} \\ \mathbf{0}' & 0 \end{pmatrix} + (1-p) \begin{pmatrix} \mathbf{0} & \pi_n \\ 0 & \mathbf{0}' \end{pmatrix} + (1-q) \begin{pmatrix} \mathbf{0}' & 0 \\ \pi_n & \mathbf{0} \end{pmatrix} + q \begin{pmatrix} 0 & \mathbf{0}' \\ \mathbf{0} & \pi_n \end{pmatrix}$$

where $\mathbf{0}$ is an n-dimensional column vector of zeros.

3. *Divide columns $2, \ldots, n$ of π_{n+1} by 2. If $n+1 = l$, stop. Otherwise, set $n = n+1$ and go to 2.*

The resulting matrix π_l of size $l \times l$ is the transition probability matrix associated to the AR(1) process. Given this transition probability matrix, the unconditional distribution of the stochastic variable converges to the stationary distribution $\lambda^{(l)} = (\lambda_1^{(l)}, \lambda_2^{(l)}, \ldots, \lambda_l^{(l)})$, where:

$$\lambda_i^{(l)} = \binom{l-1}{i-1} s^{i-1}(1-s)^{l-1}$$

where $s = \frac{1-p}{2-(p+q)}$. Given this closed-form solution for the invariant distribution, we can compute analytically the moments of the model, associated with π_l.

3.9 Stationary Distribution

In order to understand how the economy would behave in the long run, we need to compute the stationary distribution, in order to be able to compute moments in the economy. It turns out there are three ways to compute the stationary distribution. The first one involves simulating the economy over a large enough number of periods, and analyzing the moments of the economy throughout time. The second one is computing a transition matrix that states the probabilities of moving from states (k, z) today to states (k', z') tomorrow, and applying it to the unconditional distribution until it converges. The third one involves obtaining the eigenvector of the transition a matrix.

3.9.1 Simulation

The purpose of this section is to simulate the behavior of the social planner after a long history of shocks, using the solution to the social planner's problem obtained through value

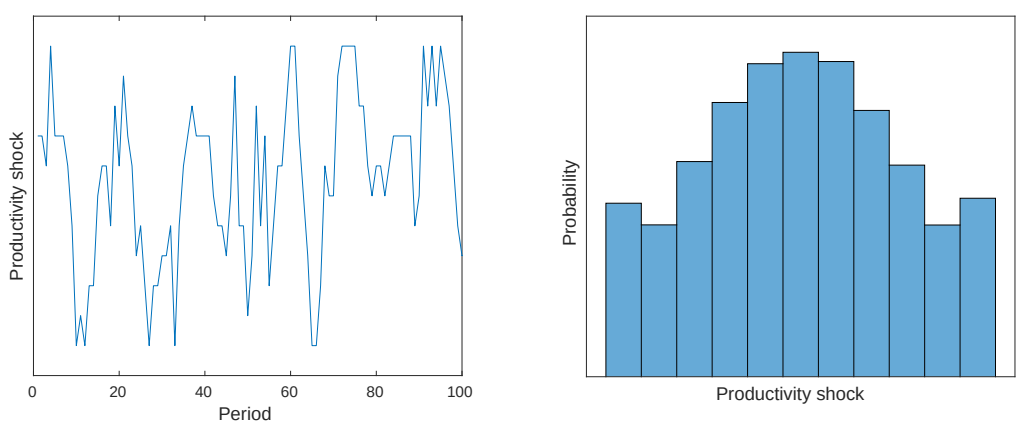
function iteration.

In order to simulate the behavior of the economy, start with a shock in period $t = 0$ given by $z \in \mathcal{Z}$. Generate a random number x from a uniform distribution $U[0, 1]$. The transition probabilities are given by $\pi(z' = z_1|z = z_i), \pi(z' = z_2|z = z_i), \ldots, \pi(z' = z_n|z = z_i)$. Set z' as:

$$z' = \begin{cases} z_1 & \text{if } x < \pi(z_1|z_i) \\ z_2 & \text{if } \pi(z_1|z_i) < x \leq \sum_{j=1}^{2} \pi(z_j|z_i) \\ z_3 & \text{if } \sum_{j=1}^{2} \pi(z_j|z_i) < x \leq \sum_{j=1}^{3} \pi(z_j|z_i) \\ \vdots & \\ z_n & \text{if } \sum_{j=1}^{n-1} \pi(z_j|z_i) < x \leq \sum_{j=1}^{n} \pi(z_j|z_i) \end{cases}$$

Repeat this process, by drawing a another random number and obtaining z'', given the z' obtained, and so on. In this way, construct a whole sequence of shocks for a sufficiently large number of periods, such as $T = 10,000$.

Panel (a) in Figure 3.3 illustrates the simulation of the shocks during 100 periods, and panel (b) illustrates the distribution of shocks after a simulation with $100,000$ periods. As can be observed, the distribution of the shocks resembles a normal distribution. This is because the transition probability matrix was obtained with Tauchen's method, by discretizing a continuous AR(1) process.



(a) Simulation of shocks.

(b) Stationary distribution in state space.

Figure 3.3: Simulation of model with exogenous productivity shocks.

Once the history of random shocks has been obtained, the next step is to compute the capital accumulation decision of the social planner. For this, assume that the economy starts off with an initial level of capital $k \in \mathcal{K}$, which could be set as the smallest value in the capital grid. Given the initial shock z and the initial level of capital k, the planner will choose $k' = k'(k, z)$. Next period, given the shock z' and capital chosen in the first period k', the planner will choose $k'' = k'(k', z')$, and so on. In a similar way, we can compute all other variables in the economy, such as consumption c, investment i and production y.

Note that we are assuming that the economy starts off with a level of capital k_0 which might not be very likely to occur in the long-run. That is, once we start our economy in period 0, it might take some periods for the economy to converge to the long-run stationary distribution. Therefore, it is recommended to discard, for example, the first $1,000$ simulations and only keep the remaining ones.

With this time series for capital and shocks, we can compute the long-run distribution along the state space, by looking at the proportion of periods the economy spent on each of point (k, z). Figure 3.4 illustrates the stationary distribution of the simulated economy. That is, it illustrates the probability that the economy is in state (k, z) at a given point in time.

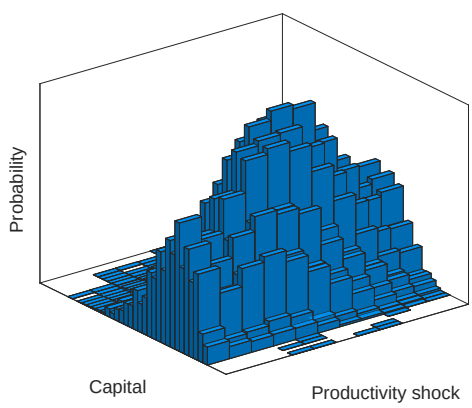


Figure 3.4: Stationary distribution of the model with exogenous productivity shocks.

Once we have computed the stationary distribution of this economy, we can compute moments, such as the standard deviation of consumption, production and investment, and correlations between variables. Table 3.1 illustrates the results for the simulation of this economy.

Moment	Value
Std. c	0.31
Std. y	1.11
Std. i	0.43
Corr(c,y)	0.81
Corr(i,y)	0.93

Table 3.1: Stationary moments of the model with exogenous productivity shocks.

In this version of the model, there is no technological growth, no population growth, and no endogenous labor.

3.9.2 Transition Matrix

The second method for computing the stationary distribution is to compute the transition matrix that, given the optimal choices of the planner, states the probability that the economy moves from state (k, z) to state (k', z'). This transition has both a deterministic and a stochastic component. Assume the planner is in state (k, z). The optimal choice of capital is dictated by its policy function $k' = k'(k, z)$, which is deterministic. However, the value z' at which the economy will appear next period is stochastic, and depends on the Markov transition probability matrix $\pi(z'|z)$. Therefore, for each $z_i \in \mathcal{Z}$, the economy will move to (k', z_i) with probability $\pi(z' = z_i|z)$, where $k' = k'(k, z)$. Similarly, the probability of moving from states (k, z) to states (k', z') for $k' \neq k'(k, z)$ is equal to zero, given that the planner will not choose that level of capital.

To compute the stationary distribution, construct the transition matrix $Q((k, z), (k', z'))$, which dictates the probability of transitioning from (k, z) to (k', z'), as follows:

$$Q((k, z), (k', z')) = \begin{cases} \pi(z'|z) & \text{if } k' = k'(k, z) \\ 0 & \text{otherwise} \end{cases}$$

Let $P_t \in R^{|\mathcal{K}| \times |\mathcal{Z}|}$ be the unconditional distribution of agents in the $\mathcal{K} \times \mathcal{Z}$ state space in period t. The evolution of the unconditional distribution is described by the following equation:

$$P_{t+1} = Q' \cdot P_t \tag{3.23}$$

where Q' is the transpose of the transition matrix Q. The stationary distribution P^* satisfies $P^* = Q' \cdot P^*$. This means that the stationary distribution P^* is the eigenvector of the transition matrix Q associated to the eigenvalue that is equal to 1. We can compute the stationary distribution of this economy simply by computing the eigenvector of Q associated to the eigenvalue equal to 1.

Similarly, note the if the economy converges to its stationary distribution, it must be the case that equation (3.23) holds in the limit. In particular:

$$P_t = Q' P_{t-1} = (Q')^2 P_{t-1} = \ldots = (Q')^t P_0$$

where $(Q')^t$ is the product of (Q') with itself t times, so the stationary distribution satisfies:

$$P^* = \lim_{t \to \infty} P_t = \lim_{t \to \infty} (Q')^t \cdot P_0$$

Therefore, to compute the stationary distribution P^*, we can start from the exogenously given unconditional distribution in period $t = 0$, and left-multiply it over and over again by Q', until convergence.

3.10 Real Business Cycles

Economic cycles have been a topic of interest for economists since the early XIX century. However, given the lack of theoretical infrastructure before the neoclassical growth model before the decade of 1960, the advances in the study of business cycles were slow. The development of the general equilibrium theory by Arrow and Debreu led macroeconomists was the first step towards the development of a macroeconomic theory with rational, maximizing agents. In his paper, Lucas (1977) defined business cycles as *"movements about trend in gross national product"*. However, he did not defined what a *trend* is.

Hodrick and Prescott (1997) proposed a procedure for obtaining the trend component that has been widely accepted in the literature. Assume we have a time series $\{Y_t\}_{t=0}^T$, where Y can be GDP or any other macroeconomic variable. We can decompose Y_t as $Y_t = Y_t^g \cdot Y_t^c$, where Y_t^g is the trend component and Y_t^c is the cyclical component. In

logarithms, $y_t = y_t^g + y_t^c$. The trend component y_t^g can be obtained as:

$$\arg\min_{y^g} \quad \sum_{t=0}^{T}(y_t - y_t^g)^2 + \lambda \sum_{t=1}^{T}\left[(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)\right]^2$$

Where λ is a parameter that determines the smoothness of the trend y_t^g. The first term $\sum_{t=0}^{T}(y_t - y_t^g)^2$ is the sum of squared deviations of the series from the trend. The second term $\sum_{t=1}^{T}\left[(y_{t+1}^g - y_t^g) - (y_t^g - y_{t-1}^g)\right]^2$ is a multiple λ of the squared differences in growth rates over time. A large value of λ penalizes more the variation in growth rates, so yields a very stable trend. In the limit, as $\lambda \to \infty$, the result is a linear trend. In the opposite case, when $\lambda = 0$, the trend is exactly equal to the original time series.

The values of λ depend on the variation we want the trend to have. Presumably, if the time series corresponds to high-frequency data, such as daily or hourly financial data, the variation in the data is large, so we should use a large value for λ. In contrast, if the frequency of the data series is small, such as yearly data, the trend can be obtained with a small value for λ. Ravn and Uhlig (2002) suggest using $\lambda = 6.24$ for yearly data, $\lambda = 1600$ for quarterly data, and $\lambda = 129,600$ for monthly data.

Once we have a procedure to obtain the trend from a time series, we can obtain the cyclical component by subtracting the trend from the time series: $y_t^c = y_t - y_t^g$. Table 3.2 provides the standard deviations of the cyclical component of consumption, investment, government purchases, exports, imports, labor supply, and capital, as well as their correlation with the cyclical component of the gross national product.

The results obtained by studying the cyclical component of aggregate variables yields the following conclusions. First, consumption, investment, and real wages are procyclical. Second, consumption, labor supply, and labor income are less volatile than the gross national product, while investment is much more volatile.

The neoclassical growth model with stochastic productivity shocks, endogenous labor supply, and exogenous technological growth is the basis of "Real Business Cycle" (RBC) theory, which studies the short run fluctuations in the economy. Cooley and Prescott (1995) calibrate the neoclassical growth model with uncertainty, endogenous labor, and exogenous productivity and population growth. Assume that population grows at a rate n, productivity grows at a rate g, households have a period utility function $u(c_t, l_t) = (1-\phi)\log(c_t) + \phi\log(1-l_t)$, and the production function is Cobb-Douglas $F(k_t, A_t l_t) =$

Variable	Std. dev. (%)	Corr. with y
Std. y	1.71	1.00
Std. c	1.25	0.82
Std. i	8.30	0.91
Std. g	2.07	0.05
Std. x	5.53	0.34
Std. m	4.92	0.71
Std. l	1.47	0.86
Std. k	0.62	-0.08
Std. wl	1.58	0.88
Std. Rk	2.93	0.84

Table 3.2: Business cycle statistics for the U.S.

$e^{z_t}(1+g)^{t(1-\alpha)}k_t^{\alpha}l_t^{1-\alpha}$. The social planner's problem is:

$$\max_{c_t,k_{t+1},l_t} \quad \mathbb{E}_0 \sum_{t=0}^{\infty} \hat{\beta}^t \left[(1-\phi)\log(c_t(z^t)) + \phi \log(1-l_t(z^t))\right] \quad s.t.$$

$$c_t(z^t) + (1+n)(1+g)k_{t+1}(z^t) = (1-\delta)k_t(z^{t-1}) + e^{z_t}(1+g)^{t(1-\alpha)}k_t(z^{t-1})^{\alpha}l_t(z^t)^{1-\alpha}$$

$$z_{t+1} = \rho z_t + \epsilon_{t+1}, \quad \epsilon_{t+1} \sim N(0,\sigma)$$

$$c_t(z^t), k_{t+1}(z^t) \geq 0, \quad k_0 \text{ given}$$

To compute this model, parameter values for $\alpha, \phi, \beta, \delta, n, g, \rho, \sigma$ must be chosen. These values are chosen such that the non-stochastic economy (that is, the economy with $z_t = 0$), matches some long-run moments in the data.

First, α is chosen, such that $1-\alpha = \frac{wl}{y}$ matches total compensation to labor. In the literature, $\alpha \approx 0.33$, but Cooley and Prescott (1995) choose $\alpha = 0.4$, because they include imputed income from government capital. Parameters n and g are set as the long-run rates of population growth and per-capita income growth, respectively. For the rest of parameters, recall that the non-stochastic version of the social planner's problem yields the

following balanced growth path conditions:

$$\frac{1}{\hat{\beta}} = \frac{1-\delta+\alpha k^{\alpha-1}l^{1-\alpha}}{(1+n)(1+g)} \quad \Longleftrightarrow \quad \frac{1}{\beta} = \frac{1-\delta+\alpha y/k}{(1+g)} \tag{3.24}$$

$$(1-\alpha)\frac{y}{c} = \frac{\phi}{1-\phi}\frac{l}{1-l} \tag{3.25}$$

$$(1+n)(1+g)k = (1-\delta)k + i \quad \Longleftrightarrow \quad \delta = \frac{i}{k} + 1 - (1+g)(1+n) \tag{3.26}$$

From equation (3.26), by using the investment rate ($i/k = 0.076$) and the values for n and g, we can obtain an estimate for annual depreciation $\delta = 0.048$, or quarterly depreciation $\delta = 0.012$. With values for δ, n, g, α, and the capital output ratio ($k/y = 3.32$), equation (3.24) pins down an annual value for $\beta = 0.947$, or a quarterly value $\beta = 0.987$. Finally, equation (3.25) pins down ϕ for a given l and y/c. The steady state labor l is set to $l = 0.31$, given that micro-estimates find that households allocate around 1/3 of their time to work. Given this, the steady-state ratio of output to consumption is $y/c = 1.33$, which yields $\phi = 0.64$.

Finally, by computing the Solow residuals for the U.S. economy, the authors find a high persistence in the productivity process. They calibrate $\rho = 0.95$ and $\sigma = 0.007$.

Moment	Model	U.S. data
Std. y	1.35	1.72
Std. c	0.33	1.27
Std. i	5.95	8.24
Std. l	0.77	1.59
Corr(y, c)	0.85	0.83
Corr(y, i)	0.99	0.91
Corr(y, l)	0.72	0.86

Table 3.3: Business cycle statistics for the U.S. and model performance

Using this calibration, Table 3.3 illustrates the moments obtained in the data for the U.S. economy and in the model. The model does a good job at replicating the direction of business cycle facts, such as the procyclicality of consumption, investment, and labor, and the magnitude of fluctuations in production, investment, and labor.

3.11 Log-Linearizing around the Steady State

Take the optimality conditions to the social planner's problem:

$$u'(c_t(z^t)) = \beta \mathbb{E}\left(1 - \delta + e^{z_{t+1}} F_k(k_{t+1}(z^t), 1)\right) u'(c_{t+1}(z^{t+1}))$$

$$c_t(z^t) + k_{t+1}(z^t) = (1-\delta)k_t(z^{t-1}) + e^{z_t} F(k_t(z^{t-1}), 1)$$

For this section, I will simplify notation and drop the dependency of variables on the shock history z^t. Moreover, assume the utility function is CRRA and the production function is Cobb-Douglas:

$$c_t^{-\sigma} = \beta \mathbb{E}\left[\left(1 - \delta + e^{z_{t+1}} \alpha k_{t+1}^{\alpha-1}\right) c_{t+1}^{-\sigma}\right] \tag{3.27}$$

$$c_t + k_{t+1} = (1-\delta)k_t + e^{z_t} k_t^{\alpha} \tag{3.28}$$

The solution to the social planner's problem are functions for consumption and capital accumulation that solve this non-linear system of equations for $t \in \{1, \ldots\}$. Solving a system of non-linear equations turns out to be much more involved that solving a system of linear equations. In this section, we will approximate this nonlinear system with a linear one, using a Taylor approximation around the non-stochastic steady state.

First, recall that in this model without uncertainty, when $z_t = 0$, the steady state of the economy is pinned down by k and c that solve:

$$1 = \beta(1 - \delta + \alpha k^{\alpha-1}) \tag{3.29}$$

$$c = k^{\alpha} - \delta k \tag{3.30}$$

Where again, to keep notation simple, as opposed to past sections, I ommited the $*$ superscritp to denote steady state variables. Define variables with a hat, as the log-deviations from steady state. For example, define:

$$\hat{c}_t = \log(c_t) - \log(c) \approx \frac{c_t - c}{c}$$

Log-deviations from steady state are approximately equal to the percentage deviation from steady state, whenever the economy is close to steady state. We can re-express the opti-

mality conditions in terms of log-deviations, by noting that $c_t = ce^{\hat{c}_t}$ and $k_t = ke^{\hat{k}_t}$:

$$e^{-\sigma\hat{c}_t} = \beta\mathbb{E}\left[\left(1-\delta+\alpha k^{\alpha-1}e^{z_{t+1}+(\alpha-1)\hat{k}_{t+1}}\right)e^{-\sigma\hat{c}_{t+1}}\right]$$

$$ce^{\hat{c}_t} + ke^{\hat{k}_{t+1}} = (1-\delta)ke^{\hat{k}_t} + k^{\alpha}e^{z_t+\alpha\hat{k}_t}$$

Let's linearize this system around $\hat{c}_t = \hat{c}_{t+1} = \hat{k}_t = \hat{k}_{t+1} = 0$, which is equivalent to linearizing the system described by equations (3.27) and (3.28) around c and k. Recall that we can approximate a function $f(x)$ around a point x_0 using a Taylor expansion of order 1, such that $f(x) \approx f(x_0) + f'(x_0)(x - x_0)$. If the function is multivarite, we can approximate $f(x, y)$ around (x_0, y_0) with $f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$. This means that:

$$\begin{aligned}
f(\hat{c}_t) &= e^{-\sigma\hat{c}_t} \approx -\sigma\hat{c}_t \\
f(\hat{z}_{t+1}, \hat{k}_{t+1}) &= e^{-\sigma\hat{c}_{t+1}}\left(1-\delta+\alpha k^{\alpha-1}e^{z_{t+1}+(\alpha-1)\hat{k}_{t+1}}\right) \\
&\approx (1-\delta+\alpha k^{\alpha-1}) - \sigma(1-\delta+\alpha k^{\alpha-1})\hat{c}_{t+1} + \\
&\qquad (1-\delta+\alpha k^{\alpha-1})z_{t+1} + (\alpha-1)(1-\delta+\alpha k^{\alpha-1})\hat{k}_{t+1} \\
&= (1-\delta+\alpha k^{\alpha-1})(-\sigma\hat{c}_{t+1} + z_{t+1} - (1-\alpha)\hat{k}_{t+1}) \\
&\vdots
\end{aligned}$$

Using these approximations, the fact that in the non-stochastic steady state equations (3.29) and (3.30) hold, and $\mathbb{E}z_{t+1} = \rho z_t$, the system of optimality conditions becomes:

$$\sigma\hat{c}_t + \rho z_t - (1-\alpha)\hat{k}_{t+1} - \sigma\mathbb{E}(\hat{c}_{t+1}) = 0 \tag{3.31}$$

$$c\hat{c}_t + k\hat{k}_{t+1} - ((1-\delta)k + \alpha k^{\alpha})\,\hat{k}_t - k^{\alpha}z_t = 0 \tag{3.32}$$

Using a result in control theory, we can solve this system and obtain policy functions that are linear functions of the state variables k_t and z_t. We will use the method of undetermined coefficients to obtain the policy functions. Write:

$$\hat{k}_{t+1}(k_t, z_t) = A\hat{k}_t + Bz_t \tag{3.33}$$

$$\hat{c}_t(k_t, z_t) = C\hat{k}_t + Dz_t \tag{3.34}$$

Now, we should plug in these policy functions in the linearized optimality conditions (3.31) and (3.32), and try to solve for A, B, C and D in terms of the parameters of the model. First, note that $\hat{c}_{t+1} = C\hat{k}_{t+1} + Dz_{t+1}$, so $\mathbb{E}\hat{c}_{t+1} = C(A\hat{k}_t + Bz_t) + D\rho z_t = AC\hat{k}_t + (BC + \rho D)z_t$. The system of linearized equations can be written as:

$$\sigma(C\hat{k}_t + Dz_t) + \rho z_t - (1-\alpha)(A\hat{k}_t + Bz_t) - \sigma(AC\hat{k}_t + (BC + \rho D)z_t) = 0$$
$$c(C\hat{k}_t + Dz_t) + k(A\hat{k}_t + Bz_t) - ((1-\delta)k + \alpha k^\alpha)\,\hat{k}_t - k^\alpha z_t = 0$$

Rearranging:

$$(\sigma C - (1-\alpha)A - \sigma AC)\,\hat{k}_t + (\sigma D + \rho - (1-\alpha)B - \sigma(BC + \rho D))\,z_t = 0$$
$$(cC + kA - (1-\delta)k - \alpha k^\alpha)\,\hat{k}_t + (cD + kB - k^\alpha)\,z_t = 0$$

Given that these equations must hold for all $\hat{k}_t$ and z_t, it must be that:

$$\sigma C - (1-\alpha)A - \sigma AC = 0 \tag{3.35}$$
$$cC + kA - (1-\delta)k - \alpha k^\alpha = 0 \tag{3.36}$$
$$\sigma D + \rho - (1-\alpha)B - \sigma(BC + \rho D) = 0 \tag{3.37}$$
$$cD + kB - k^\alpha = 0 \tag{3.38}$$

These are two systems of equations in two unknowns each. We can solve this system to obtain the values of A, B, C and D in terms of the model parameters, to construct the linearized policy functions $\hat{k}_{t+1}(\hat{k}_t, z_t)$ and $\hat{c}_t(\hat{k}_t, z_t)$, characterized by equations (3.33) and (3.34). Note that the constants A and C can be obtained by solving the system composed only by equations (3.35) and (3.36). Once we have solved for C, we can solve the system of equations (3.37) and (3.38), to obtain B and D.

Moreover, note that by solving for C in (3.36) and plugging in (3.35), we obtain:

$$[(1-\delta)k + \alpha k^\alpha - Ak] - \left[(1-\alpha) + \frac{\sigma}{c}(1-\delta)k + \frac{\sigma}{c}\alpha k^\alpha\right] A + \frac{\sigma k}{c}A^2 = 0 \tag{3.39}$$

Which is a quadratic equation on A, with two roots. One of these solutions is such that $A > 1$ and the other one satisfies $A < 1$. However, only the solution for which $A < 1$ will satisfy the transversality condition of the social planner's problem. This is because, for the

value $A > 1$, the policy function at $\hat{k}_t$ and $z_t = 0$ is $\hat{k}_{t+1} = A\hat{k}_{t+1}$. This means that any deviation of k_t from steady state ($\hat{k}_t > 0$) will generate an even larger deviation in $t+1$, and so on. This means that deviations from steady state would generate diverging behavior in the long run, violating the transversality condition.

Once a solution is obtained, the corresponding policy functions are:

$$k_{t+1} = k\left(\frac{k_t}{k}\right)^A e^{Bz_t}$$

$$c_t = k\left(\frac{k_t}{k}\right)^C e^{Dz_t}$$

Figure 3.5 illustrates the policy functions for the value function iteration algorithm as well as for the log-linearization procedure. Around the steady state value for capital, the log-linear approximation is very close to the optimal capital accumulation decision, given by the policy function of the value function iteration. However, in regions of the state space that are far from the steady state, the log-linear approximation is not very precise.

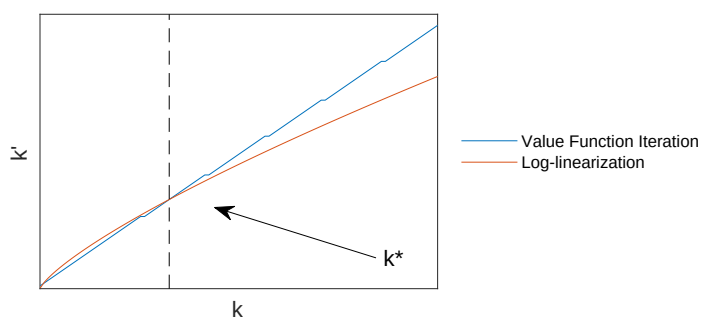


Figure 3.5: Capital policy function for value function iteration and log-linearization.

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