Midterm Exam

Dynamic Macroeconomics I

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1 Sequential Formulation (50 points)

Assume a representative agent economy with endogenous labor supply, no population growth ($n = 0$), and no technological growth ($g = 0$). Households value leisure and consumption, according to a lifetime utility function: $\sum_{t=0}^{\infty} \beta^t u(c_t, 1-l_t)$, where l_t is the amount of labor supplied. Capital depreciates at a constant rate $\delta > 0$, and there is a government that has expenditures equal to g_t at period t, where $g_{t+1} = \rho g_t$, g_0 is given and $\rho \leq 1$. The resource constraint is:

$$c_t + k_{t+1} + g_t = (1-\delta)k_t + F(k_t, l_t)$$

Assume that households and the social planner take expenditures $\{g_t\}_{t=0}^{\infty}$ as given and beyond their control.

a. **Social planner:** State the social planner's problem sequentially and solve it to characterize the Pareto-optimal allocations.

The social planner's problem is:

$$\max_{c_t, l_t, k_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t, 1-l_t) \quad s.t.$$

$$c_t + k_{t+1} + g_t = (1-\delta)k_t + F(k_t, l_t)$$

$$g_{t+1} = \rho g_t, \quad c_t, k_{t+1} \geq 0, \quad k_0, g_0 \text{ given}$$

The optimality conditions are, for all $t \geq 0$:

Euler eq.: $$\frac{u_c(c_t, 1-l_t)}{\beta u_c(c_{t+1}, 1-l_{t+1})} = 1 - \delta + F_k(k_{t+1}, l_{t+1})$$

Intratemporal: $$u_c(c_t, 1-l_t)F_l(k_t, l_t) = u_l(c_t, 1-l_t)$$

Res. const.: $$c_t + k_{t+1} + g_t = (1-\delta)k_t + F(k_t, l_t)$$

Transv. cond.: $$\lim_{t\to\infty} \beta^t u_c(c_t, 1-l_t)k_{t+1} = 0$$

Init. cond.: k_0 given

b. **Competitive equilibrium:** To finance expenditures g_t, assume there is a government that charges labor income taxes τ_t^l, such that after-tax income of households is $(1-\tau_t^l)w_t l_t$, where w_t is the wage. Also, the government levies consumption taxes τ_t^c, such that total expenditures on consumption are equal to $(1+\tau_t^c)c_t$. Note that τ_t^c and τ_t^l can be negative, in which case they are interpreted as subsidies.

Assume that every period the government chooses τ_t^l and τ_t^c such that its budget is balanced in equilibrium, which means that:

$$g_t = \tau_t^l w_t l_t + \tau_t^c c_t$$

In addition, households have access to risk-free bonds that yield net returns equal to r_t. Define a competitive equilibrium.

The household's problem is:

$$\max_{c_t, l_t, k_{t+1}, a_{t+1}} \quad \sum_{t=0}^{\infty} \beta^t u(c_t, 1-l_t^s) \quad s.t.$$

$$(1+\tau_t^c)c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + R_t k_t + (1-\tau_t^l)w_t + (1+r_t)a_t$$

$$a_{t+1} \geq -\bar{A}$$

$$c_t, k_{t+1} \geq 0, \quad k_0, a_0 = 0 \text{ given}$$

The firm's problem is:

$$\max_{k_t, l_t} \quad F(k_t, l_t) - w_t l_t - R_t k_t$$

A competitive equilibrium are allocations for the household $\{c_t, l_t^s k_{t+1}^s, a_{t+1}\}_{t=0}^{\infty}$, allocations for the firm $\{k_t^d, l_t^d\}_{t=0}^{\infty}$, prices $\{w_t, R_t, r_t\}_{t=0}^{\infty}$, and taxes $\{\tau_t^l, \tau_t^c\}_{t=0}^{\infty}$, such that:

(a) $\{c_t, l_t^s k_{t+1}^s, a_{t+1}\}_{t=0}^{\infty}$ solve the household's problem.

(b) $\{k_t^d, l_t^d\}_{t=0}^{\infty}$ solve the firm's problem

(c) $\{w_t, R_t, r_t\}_{t=0}^{\infty}$ are such that markets clear:

   i. Goods: $c_t + k_{t+1} + g_t = (1-\delta)k_t + F(k_t, l_t)$

   ii. Labor: $l_t^d = l_t^s$

   iii. Capital: $k_t^d = k_t^s$

   iv. Assets: $a_t = 0$

(d) $\{\tau_t^l, \tau_t^c\}_{t=0}^{\infty}$ are such that the government's budget is balanced:

$$g_t = \tau_t^l w_t l_t + \tau_t^c c_t$$

c. Solve the household's and firm's problem and characterize a competitive equilibrium.

The household's optimality conditions are, for all $t \geq 0$:

Euler eq.: $$\frac{u_c(c_t, 1-l_t)}{\beta u_c(c_{t+1}, 1-l_{t+1})} \cdot \left(\frac{1+\tau_{t+1}^c}{1+\tau_t^c}\right) = 1 - \delta + R_{t+1}$$

$$\frac{u_c(c_t, 1-l_t)}{\beta u_c(c_{t+1}, 1-l_{t+1})} \cdot \left(\frac{1+\tau_{t+1}^c}{1+\tau_t^c}\right) = 1 + r_{t+1}$$

Intratemporal: $$w_t u_c(c_t, 1-l_t) \cdot \left(\frac{1-\tau_t^l}{1+\tau_t^c}\right) = u_l(c_t, 1-l_t)$$

Budget const.: $$(1+\tau_t^c)c_t + k_{t+1} + a_{t+1} = (1-\delta)k_t + R_t k_t + (1-\tau_t^l)w_t + (1+r_t)a_t$$

Transv. cond.: $$\lim_{t\to\infty} \frac{\beta^t u_c(c_t, 1-l_t)}{1+\tau_t^c} k_{t+1} = 0$$

$$\lim_{t\to\infty} \frac{\beta^t u_c(c_t, 1-l_t)}{1+\tau_t^c} a_{t+1} = 0$$

Init. cond.: $k_0, a_0 = 0$ given

The firm's optimality conditions are:

$$w_t = F_l(k_t, l_t)$$
$$R_t = F_k(k_t, l_t)$$

d. State the conditions under which there is a steady state and characterize steady state allocations.

To characterize the steady state, we can use the social planner's problem. In steady state, $c_t = c^*, l_t = l^*, k_t = k^*, g_t = g^*, w_t = w^*, R_t = R^*, r_t = r^*, \tau_t^c = \tau_c^*, \tau_t^l = \tau_l^*$. The only way in which g_t can remain constant is if $\rho = 1$, in which case $g^* = g_0$. So there is a steady state only if $\rho = 1$. The steady state is characterized by $c^*, l^*, k^*, w^*, R^*, r^*, \tau_c^*, \tau_l^*$ that solve:

$$\frac{1}{\beta} = 1 - \delta + F_k(k^*, l^*)$$
$$u_c(c^*, 1 - l^*)F_l(k^*, l^*) = u_l(c^*, 1 - l^*)$$
$$c^* + k^* + g_0 = (1 - \delta)k^* + F(k^*, l^*)$$
$$w^* = F_l(k^*, l^*)$$
$$R^* = F_k(k^*, l^*)$$
$$r^* = R^* - \delta$$
$$g^* = \tau_c^* c^* + \tau_l^* w^* l^*$$

Note that there are infinite combinations of taxes that satisfy these equations.

e. Is the competitive equilibrium Pareto optimal? You don't need to provide a formal proof, but use the answers above to explain your answer and give intuition. If the competitive equilibrium is not Pareto optimal, which policy could the government implement for the equilibrium to be Pareto optimal?

For any set of taxes τ_t^c and τ_t^l, it is easy to see that the allocations that satisfy the social planner's optimality conditions do not satisfy the competitive equilibrium conditions. Therefore, the competitive equilibrium is not Pareto optimal. The reason is that the labor income and consumption taxes distort the intertemporal and intratemporal decisions of households. Taxes to consumption and labor distort the relative prices of c_t and l_t, generating a distortion in the intratemporal (labor vs. consumption) margin. Similarly, different taxes to consumption over time distort the intertemporal (present vs. future consumption) decision.

The only way in which the government could achieve Pareto-optimal allocations is if it sets taxes in such a way that the optimality conditions of the social planner and competitive equlibrium are the same. This means that:

$$\left(\frac{1+\tau_{t+1}^c}{1+\tau_t^c}\right) = 1, \quad \left(\frac{1-\tau_t^l}{1+\tau_t^c}\right) = 1, \quad g_t = \tau_t^c c_t + \tau_t^l w_t l_t$$

This can happen if, and only if, $\tau_{t+1}^c = \tau_t^c$, $\tau_t^c = -\tau_t^l$, and $g_t = \tau_t^c(c_t - w_t l_t)$. Only if these conditions hold, the intertemporal and intratemporal decisions of households are not distorted and the social planner can achieve the Pareto-optimal allocations.

2 Recursive Formulation (30 points)

Assume the same environment as in Question 1.

a. State the social planner's problem recursively. What are the state and control variables?

The state variables for the social planner are k and g and the control variables are c, l, and k'. The recursive formulation is:

$$V(k,g) = \max_{c,l,k'} \quad u(c, 1-l) + \beta V(k', \rho g), \quad s.t.$$

$$c + k' + g = (1-\delta)k + F(k,l)$$

b. Does a solution to the recursive formulation of the social planner's problem exist? Is it unique? Explain what conditions ensure the existence and uniqueness and show that the social planner's problem satisfies these conditions.

Yes, the social planner's problem has a unique solution. Define the operator $T : B(\mathbb{R}_+^2) \to B(\mathbb{R}_+^2)$, that maps the set of bounded functions over $\mathbb{R}_+^2$ into itself, such that:

$$T(f(x,y)) = \max_{l,x'} \quad u((1-\delta)k + F(k,l) - x' - y, 1-l) + \beta f(x', \rho y)$$

Blackwell's conditions are sufficient for a function to be a contraction mapping. If we can show that T satisfies Blackwell's conditions, then T is a contraction mapping, so by the contraction mapping theorem it has a unique fixed point. This fixed point is the solution to the social planner's problem.

(a) Monotonicity: Take $f, h \in B(\mathbb{R}^2)$, such that $f(x,y) \leq h(x,y)$ for all $(x,y) \in \mathbb{R}^2$.

$$\begin{aligned}
T(f(x,y)) &= \max_{l,x'} \quad u((1-\delta)k + F(k,l) - x' - y, 1-l) + \beta f(x', \rho y) \\
&= u((1-\delta)k + F(k,l) - g_{f,k}(x,y) - y, 1 - g_{f,l}(x,y)) + \beta f(g_{f,k}(x,y), \rho y) \\
&\leq u((1-\delta)k + F(k,l) - g_{f,k}(x,y) - y, 1 - g_{f,l}(x,y)) + \beta h(g_{f,k}(x,y), \rho y) \\
&\leq \max_{l,x'} \quad u((1-\delta)k + F(k,l) - x' - y, 1-l) + \beta h(x', \rho y) \\
&= T(h(x,y))
\end{aligned}$$

(b) Discounting:

$$\begin{aligned}
T(f(x,y)+a) &= \max_{l,x'} \quad u((1-\delta)k + F(k,l) - x' - y, 1-l) + \beta \left(f(x', \rho y) + a\right) \\
&= \max_{l,x'} \quad u((1-\delta)k + F(k,l) - x' - y, 1-l) + \beta f(x', \rho y) + \beta a \\
&= T(f(x,y)) + \beta a
\end{aligned}$$

Then, T is a contraction mapping with modulus β, so there exists a unique function V, such that $T(V) = V$. This means that the social planner's problem has a unique solution.

c. Define a recursive competitive equilibrium. Be sure to clearly identify what are the state and control variables of the household.

The individual state variables for the household are k and a. The aggregate state variables are K, L and G. The control variables are c, l, k', a'. The recursive formulation of the household's problem is:

$$V(a,k,K,G) = \max_{c,l,k',a'} \quad u(c,1-l) + \beta V(a',k',K',\rho G), \quad s.t.$$

$$(1+\tau_c(K,G))c + k' + a' = (1-\delta)k + R(K,G)k + (1-\tau_l(K,G))w(K,G)l + (1+r(K,G))a$$

$$K' = H(K,G)$$

A recursive competitive equilibrium are a value function $V(a,k,K,G)$, policy functions $k'(a,k,K,G), l(a,k,K,G), c(a,k,K,G)$, pricing functions $w(K,G), R(K,G), r(K,G)$, tax functions $\tau^c(K,G), \tau^l(K,G)$, and an aggregate law of motion $H(K,G)$, such that:

(a) $V(a,k,K,G)$ solves the household's problem, with $k'(a,k,K,G), l(a,k,K,G), c(a,k,K,G)$ being the associated policy functions

(b) The pricing functions are such that:

$$w(K,G) = F_l(K, l(0,K,K,G))$$
$$R(K,G) = F_k(K, l(0,K,K,G))$$
$$r(K,G) = R(K,G) - \delta$$

(c) The tax functions are such that the government has a balanced budget:

$$G = \tau^c(K,G)c(0,K,K,G) + \tau^l(K,G)w(K,G)l(0,K,K,G)$$

(d) Markets clear:

$$c(0,K,K,G) + k'(0,K,K,G) + G = (1-\delta)K + F(K, l(0,K,K,G))$$

(e) Consistency:

$$H(K,G) = k'(0,K,K,G)$$