Final Exam

Dynamic Macroeconomics I

Professor: David Zarruk

ITAM

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1 Sequential Formulation (50 points)

Assume an Aiyagari-type heterogeneous agent economy in which there is a continuum of households, indexed by $i \in [0, 1]$. In period $t = 0$, agents choose the streams of consumption c_t and assets a_{t+1} that maximize their expected lifetime utility:

$$\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$$

Every period, agents have one unit of time that they supply inelastically as labor, and face non-insurable idiosyncratic productivity shocks λ_t^i, such that their labor income is equal to $\lambda_t^i w_t$. Assume that $\lambda_t \in \Lambda$, and the possible set of realizations is finite: $|\Lambda| < \infty$. Households are ex-ante identical, but their histories $\lambda^{i,t} = (\lambda_1^i, \lambda_2^i, \ldots, \lambda_t^i)$ differ according to the realization of shocks. The probability that history $\lambda^{i,t}$ is realized is given by $\pi(\lambda^{i,t})$.

Assume also that markets are incomplete, so households only have access to a non-state contingent bond a_{t+1} that pays an interest r_t. Households choose a_{t+1} subject to an exogenous borrowing constraint $a_{t+1} \geq -\bar{A}$.

The final good is produced using capital K_t and labor L_t, according to a production function $F(K_t, L_t)$.

a. State the household's problem in sequential formulation and characterize the optimal allocations. In equilibrium, what are the optimal allocations functions of? *Briefly* discuss.

$$\mathbb{E}\sum_{t=0}^{\infty}\beta^t u(c_t^i)=\sum_{t=0}^{\infty}\sum_{\lambda^{i,t}\in\Lambda^t}\beta^t\pi(\lambda^{i,t})u(c_t^i)\quad s.t.$$

$$c_t^i(\lambda^{i,t})+a_{t+1}^i(\lambda^{i,t})=(1+r_t)a_t^i(\lambda^{i,t-1})+\lambda_t^i w_t$$

The optimality conditions are:

$$u'(c_t^i(\lambda^{i,t}))=\beta(1+r_t)\mathbb{E}u'(c_{t+1}(\lambda^{i,t+1}))$$
$$c_t^i(\lambda^{i,t})+a_{t+1}^i(\lambda^{i,t})=(1+r_t)a_t^i(\lambda^{i,t-1})+\lambda_t^i w_t$$

where the expectation is taken with respect to the possible realizations of $\lambda^{i,t+1}$, given $\lambda^{i,t}$. The Euler equation can be rewritten as:

$$u'(c_t^i(\lambda^{i,t}))=\beta(1+r_t)\sum_{\lambda^{i,t+1}\in\Lambda}\left(\frac{\pi(\lambda^{i,t+1})}{\pi(\lambda^{i,t})}\right)u'(c_{t+1}(\lambda^{i,t+1}))$$

The optimal allocations are functions of the entire history of shocks $\lambda^{i,t}$, rather than only on the realization of period t's shock λ_t^i. The reason is that the whole history of shocks determines the savings decision of households over time, so a_t^i depends on the whole history. Given that c_t^i depends on total savings of the household a_t^i, then c_t^i is also a function of the whole history.

b. Assume now that markets are **complete**. What assets should be added to this environment? Write the household's problem and find the optimality conditions.

In a setting with complete markets, households have access to state-contingent bonds. This means that agents can buy assets that pay only in every single realization of the idiosyncratic shock in the future. Therefore, we must add to this environment state-contingent bonds $b(\lambda^{i,t},\lambda_{t+1}^i)$ with price $q(\lambda^{i,t},\lambda_{t+1}^i)$, that agents buy after history $\lambda^{i,t}$ and pay one unit in state λ_{t+1}^i. The household's problem is:

$$\sum_{t=0}^{\infty}\sum_{\lambda^{i,t}\in\Lambda^t}\beta^t\pi(\lambda^{i,t})u(c_t^i)\quad s.t.$$

$$c_t^i(\lambda^{i,t})+\sum_{\lambda^{i,t+1}\in\Lambda}q_t(\lambda^{i,t},\lambda_{t+1}^i)b_{t+1}^i(\lambda^{i,t},\lambda_{t+1}^i)=(1+r_t)b_t^i(\lambda^{i,t})+\lambda_t^i w_t$$

The optimality conditions are:

$$u'(c_t^i(\lambda^{i,t})) = \beta \left(\frac{1}{q(\lambda^{i,t}, \lambda_{t+1}^i)}\right) \left(\frac{\pi(\lambda^{i,t+1})}{\pi(\lambda^{i,t})}\right) u'(c_{t+1}(\lambda^{i,t+1}))$$

$$c_t^i(\lambda^{i,t}) + a_{t+1}^i(\lambda^{i,t}) = (1 + r_t)a_t^i(\lambda^{i,t-1}) + \lambda_t^i w_t$$

Note that in this environment, there is no expectation in the Euler equation because households can smooth consumption over time and over realizations of the shock.

c. Are the equilibrium allocations of the complete and/or incomplete markets environments Pareto-optimal? *Briefly* explain.

Only the allocations of the complete markets environment are Pareto-optimal. Those of the incomplete markets are not. In the incomplete markets case, given the lack of insurance, households over-accumulate assets, so total capital in the economy is beyond the efficient level. In contrast, in the complete markets case, households can perfectly insure against negative productivity shocks and the allocations are the same as in the social planner's problem.

d. Assume markets are incomplete and there is a government that observes the realization of histories $\lambda^{i,t}$ for all $i \in [0, 1]$. The government can impose state contingent taxes $\tau(\lambda^{i,t})$ on asset returns. This means that the returns to a household with asset holdings a_t are $(1+(1-\tau(\lambda^{i,t}))r_t)a_t$. Can the government choose $\tau(\lambda^{i,t})$ such that the competitive equilibrium allocations are Pareto-optimal?

Yes, the government could choose the state-contingent taxes to be such that consumption is smoothed over time and states. Taxes would be such that:

$$(1 + (1 - \tau(\lambda^{i,t+1}))r_t) = \left(\frac{1}{q(\lambda^{i,t}, \lambda_{t+1}^i)}\right)$$

$$\iff \quad \tau(\lambda^{i,t+1}) = 1 - \left(\frac{1}{q(\lambda^{i,t}, \lambda_{t+1}^i)} - 1\right) \cdot \left(\frac{1}{r_{t+1}}\right)$$

With these taxes, the Euler equation that relates consumption $c_t^i(\lambda^{i,t})$ with consumption in every single state in $t + 1$, $c_t^i(\lambda^{i,t+1})$, would hold with equality, as in the complete markets case.

Note that with a state-contingent tax, the social planner can generate state contingent after-tax returns on the asset a_t, so it can correct the existing distortion.

2 Recursive Formulation (30 points)

Assume the same environment as in Question 1.d., but now the productivity shocks λ_t^i follow a Markov process of order 1, where the transition probability is denoted as $\pi(\lambda_{t+1}^i|\lambda_t^i)$. In this environment, assume the government levies a tax on asset returns that only depends on the current realization of the shock: $\tau(\lambda_t^i)$.

a. State the household's problem in recursive formulation. What are the state variables?

The individual state variables are a and λ. The aggregate state variable is the distribution of agents μ over the state space $[-\bar{A}, \infty) \times \Lambda$. The household's problem is:

$$
\begin{aligned}
V(a, \lambda, \mu) \quad = \quad & \max_{c,a'} \quad u(c) + \beta \mathbb{E} V(a', \lambda', \mu') \quad s.t. \\
& c + a' = (1 + (1 - \tau(\lambda))r(\mu))a + \lambda w(\mu) \\
& \lambda' \sim \pi(\lambda'|\lambda) \\
& \mu' = \Gamma(\mu)
\end{aligned}
$$

where Γ is an aggregate law of motion of the distribution of agents.

b. Define a recursive competitive equilibrium.

A recursive competitive equilibrium are a value function $V(a, \lambda, \mu)$, policy functions $c(a, \lambda, \mu)$ and $a'(a, \lambda, \mu)$, pricing functions $w(\mu), r(\mu)$, aggregate demands for capital K and labor L, government taxes $\tau(\lambda)$, and an aggregate law of motion Γ, such that:

(a) $V(a, \lambda, \mu)$, $c(a, \lambda, \mu)$ and $a'(a, \lambda, \mu)$ solve the household's problem.

(b) $w(\mu) = F_l(K, L)$ and $r(\mu) = F_k(K, L) - \delta$.

(c) Markets clear:

$$\int (c(a,\lambda,\mu) + a'(a,\lambda,\mu))\, d\mu(a,\lambda) = (1-\delta)K + F(K,1)$$

$$\int a'(a,\lambda) d\mu(a,\lambda) = K$$

$$\int \lambda d\mu(a,\lambda) = L = 1$$

(d) Government's budget balance:

$$\int \tau(\lambda) r(\mu) d\mu(a,\lambda) = 0$$

(e) The aggregate law of motion is consistent with household's policy functions.

c. Describe an algorithm to compute a *stationary* competitive equilibrium. (You don't have to describe how to solve the household's problem in here).

In a stationary competitive equilibrium, the distribution of agents is constant, so aggregate capital and prices are constant. The algorithm to solve for the equilibrium is as follows:

(a) Set $n = 0$ and start with a guess for aggregate capital K_0.

(b) For $n \geq 0$, set prices $w_n = F_l(K_n, 1)$ and $r_n = F_k(K_n, 1) - \delta$.

(c) Given w_n and r_n, solve the household's problem to obtain the policy function for assets $a'(a,\lambda)$.

(d) Given the policy functions and w_n and r_n, compute the stationary distribution μ_n.

(e) Compute total assets in the economy:

$$A = \int a'(a,\lambda) d\mu_n(a,\lambda)$$

(f) If $|A - K_n| < \epsilon$, stop. Otherwise, set $K_{n+1} = (A + K_n)/2$, $n = n + 1$ and go to (b).

3 Wealth Inequality (20 points)

Wealth concentration has dramatically increased during the last decades in the U.S. In light of the papers studied in class, explain what are the main reasons for the increase in wealth concentration. What are the main ingredients of a model to study these mechanisms?

See Kaymak and Poschke's paper.