ECON 897 Test (Week 4)
Aug 5, 2016

Important: This is a closed-book test. No books or lecture notes are permitted. You have **100** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

1. **(15 points)** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ and $h : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by:

$$f(x,y) = (xy, x\cos y, x\sin y)$$

$$g(x,y,z) = 2x^2y + e^{yz} + zx$$

$$h(x,y,z) = \begin{cases} z \cdot \left(\frac{x^2y}{x^2+y^2}\right) & \text{if } x, y \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

   (a) Are the functions f, g and h differentiable? Be sure to say why they are or why they are not.

   (b) Define $s = g \circ f$. Find the representation matrix of $(Ds)_{(x,y)}$.

   (c) Do the representation matrices of $(D^2f)_{(x,y)}$ and $(D^2g)_{(x,y,z)}$ exist? If they do, find them.

2. **(20 points)** A firm currently produces a good according to the production:

$$f(k,l) = Ak^{\alpha}l^{1-\alpha}$$

   The factors of production k and l are hired at market prices r and w, respectively. The firm chooses the amounts of labor and capital hired that maximize profits, which are given by the first order conditions:

$$\alpha PAk^{\alpha-1}l^{1-\alpha} = r, \qquad (1-\alpha)PAk^{\alpha}l^{-\alpha} = w$$

   (a) Compute the hessian of f.

   (b) Find expressions for the derivatives of k and l with respect to the prices P, w and r? State clearly any assumptions we have to make about f?

3. **(20 points)** Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ such that $f(x) = \frac{h(x)}{g(x)}$ and $h(x), g(x) > 0$ for all $x \in \mathbb{R}^n_+$. Assume h is concave and g is convex. Show that f is quasiconcave.

4. **(20 points)** Let $f : X \to \mathbb{R}$, where $X \subseteq \mathbb{R}^n$ non-empty and convex set. Assume f is strictly quasiconcave. Show that any local maximum is a global maximum.

5. **(25 points)** Suppose $f : (a, b) \to \mathbb{R}$ is concave.

   (a) Fix any x_0 and define

   $$g(x) = \frac{f(x) - f(x_0)}{x - x_0} \quad \forall x \in (a, b)\backslash\{x_0\}.$$

   Prove $g(x) \geq g(x')$ when $x < x'$.

   (b) Use part (a) to prove f is continuous. (Hint: the domain is open and you may want to prove $\lim_{x \to x_0} \big[f(x) - f(x_0)\big] = 0$ for all x_0.)