ECON 897 Test (Week 4)
Aug 7, 2015

Important: This is a closed-book test. No books or lecture notes are permitted. You have **120** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

If you use want to use a theorem that we proved in class, be sure to say exactly which theorem you are using, state all of its assumptions and be sure that they are satisfied. Otherwise, you will be given partial credit.

1. **(15 points)** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$, and $g : \mathbb{R}^3 \longrightarrow \mathbb{R}$ defined by:

$$g(x, y, z) = xy + yz + zx$$
$$f(x, y) = (xy, x\cos y, x\sin y)$$

   (a) **(5 points)** Are the functions f and g differentiable? Be sure to say why they are or why they aren't.

   (b) **(5 points)** Define $h = g \circ f$. Find the representation matrix of $(Dh)_{(x,y)}$.

   (c) **(5 points)** Do the representation matrices of $(D^2 f)_{(x,y)}$ and $(D^2 g)_{(x,y,z)}$ exist? If they do, find them.

2. **(20 points)** Suppose there are n goods. To each price vector $p = (p_1, \cdots, p_n) \in \mathbb{R}^n_{++}$ corresponds a unique demand vector $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_{++}$ implicitly defined by the following n equations:

$$\begin{aligned} U_1(x_1, \cdots, x_n) &= p_1 \\ \vdots \quad & \vdots \quad \vdots \\ U_n(x_1, \cdots, x_n) &= p_n \end{aligned}$$

   where $U = (U_1, \cdots, U_n) : \mathbb{R}^n_{++} \to \mathbb{R}^n_{++}$ is continuously differentiable and the representation matrix of DU is a negative definite matrix.

   (a) **(10 points)** Prove that if the price of ith good increases, then the demand for this good decreases, i.e. $\frac{\partial x_i}{\partial p_i} < 0$, $1 \leq i \leq n$. (Recall a symmetric matrix A is negative definite if for all $x \neq 0$, $x^T Ax < 0$.)

   (b) **(10 points)** If $n = 2$, find expressions for $\frac{\partial x_i}{\partial p_j}$, for $i, j \in \{1, 2\}$.

3. **(15 points)** One of the separating hyperplane theorems that we proved was: Let $D \subseteq \mathbb{R}^n$ be compact and convex, and $E \subseteq \mathbb{R}^n$ be closed and convex. Assume $D \cap E = \emptyset$. Then, there exists a hyperplane $H(p, a)$ such that $p \cdot e < a$ for all $e \in E$ and $p \cdot d > a$ for all $d \in D$. Give an example of a case in which the set D is not compact and, therefore, there does not exist such a hyperplane.

4. **(20 points)** Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be homogeneous of degree 1 and quasi-concave. Prove that f is concave.

   Remember that f is homogeneous of degree 1 if $f(\lambda x) = \lambda f(x)$, for all $\lambda \neq 0$.

5. **(30 points)** Let $U : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$ be continuous, quasi-concave and increasing[1]. Let $x^* \in \mathbb{R}^n_{++}$.

   (a) **(10 points)** Prove that there exists a $p \in \mathbb{R}^n$ and $M \in \mathbb{R}$ such that $p \cdot x^* \leq M$ and $p \cdot x \geq M$ for all x such that $U(x) \geq U(x^*)$.

   (b) **(10 points)** Prove that $p \cdot x > M$ if x is such that $U(x) > U(x^*)$.

   (c) **(10 points)** Prove that, in fact, $p \in \mathbb{R}^n_+$.

[1] Definition: A function $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ is increasing if $F(x) > F(y)$ whenever $x >> y$, where $x >> y$ means that $x_i > y_i$ for all $i \in \{1, \ldots, n\}$.