ECON 897 Test (Week 3)
July 29, 2016

Important: This is a closed-book test. No books or lecture notes are permitted. You have **120** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

1. **(10 points)** State whether the following function is twice continuously differentiable, or not:

$$f(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0 \end{cases}$$

Proof. It is not. For $x \geq 0$, $f'(x) = 2x$, while for $x < 0$, $f'(x) = -2x$. This means that $f'(x) = 2|x|$, which is not differentiable. Therefore, the second derivative of f does not exist at $x = 0$. □

2. **(10 points)** Does there exist a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(0) = 0$ and $f'(x) \geq 1$ for all $x \neq 0$? If it does, show an example. If not, prove it.

Proof. There does not exist. Assume it does. Let $x \neq 0$. By MVT:

$\frac{f(x)-f(0)}{x-0} = f'(\zeta) \geq 1, \quad \zeta \in (0, x)$. Taking limits,

$f'(0) = \lim_{x\to 0} \frac{f(x)-f(0)}{x} \geq 1$ which is a contradiction. □

3. **(30 points)**

   (a) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f'(x)$ exist and are finite, then $\lim_{x\to\infty} f'(x) = 0$.

   Proof. Assume $\lim_{x\to\infty} f'(x) = c > 0$. Let $\epsilon = c/2$. There exists R, such that $|f'(x) - c| < \epsilon, \forall x > R$. This means that $f'(x) > c - \epsilon, \forall x > R$.

Let x_0, x such that $R < x_0 < x$. By MVT:

$$f(x) - f(x_0) = f'(\theta)(x - x_0), \quad \theta \in (x_0, x)$$

$$f(x) = f(x_0) + f'(\theta)(x - x_0) > f(x_0) + (c - \epsilon)(x - x_0) = \tfrac{c}{2}(x - x_0) + f(x_0)$$

Since $c \neq 0$, $f(x) \to \infty$ as $x \to \infty$, which contradicts the fact that $\lim_{x\to\infty} f(x)$ exits and is finite. ☐

(b) Prove that if $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} f''(x)$ exist, then $\lim_{x\to\infty} f''(x) = 0$.

Proof. Assume $\lim_{x\to\infty} f''(x) = c > 0$. By the same argument as above, there exists R_0 such that $\forall x, x_0 > R_0$

$$f'(x) > \tfrac{c}{2}(x - x_0) + f'(x_0)$$

This means that for some R_1 sufficiently large, $f'(x) > 0, \forall x > R_1$. Let $x_0, x_1 > \max\{R_0, R_1\}$. Using again the MVT,

$$f(x_1) - f(x_0) = f'(\theta)(x_1 - x_0), \quad \theta > x_0 > \max\{R_0, R_1\}$$

$$f(x_1) = f(x_0) + f'(\theta)(x_1 - x_0), \quad \forall x_1 > \max\{R_0, R_1\}.$$

This contradicts that $\lim_{x\to\infty} f(x)$ exits and is finite. ☐

4. **(15 points)** State whether the following are linear subspaces, and prove your answer:

(a) Let $W_n = \{f(x) \in P(F) | f(x) = 0$ or f(x) has degree exactly equal to $n > 1\}$. Is W_n a subspace of $P(F)$, where $P(F)$ is the space of polynomials?

Proof. It is not a subspace:

Take two polynomials of degree n with the same leading coefficient and subtract them. The resulting polynomial has degree $n-1$, so W_n is not closed under addition. For example, take:

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in W_n$$

$$x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0 \in W_n$$

Subtracting them, it is clear that

$$(a_{n-1} - b_{n-1})x^{n-1} + \ldots + (a_1 - b_1)x + (a_0 - b_0) \notin W_n$$

□

(b) Let $A = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = a_3 + 2\}$. Is A a subspace of $\mathbb{R}^3$?

Proof. It is not a subspace:

It is not closed under addition. Take $(3, 0, 1) \in A$ and $(5, 0, 3) \in A$. Clearly, $(3, 0, 1) + (5, 0, 3) = (8, 0, 4) \notin A$.

□

5. **(15 points)** Consider P_2, the space of polynomials of degree 2. Let $a, b, c \in \mathbb{R}$, and define:

$$f_1(x) = \frac{(x-a)(x-b)}{(c-a)(c-b)}, \quad f_2(x) = \frac{(x-c)(x-a)}{(b-c)(b-a)}, \quad f_3(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)}$$

(a) Define what a basis of a subspace is.

(b) Prove that $\mathbb{F} = \{f_1(x), f_2(x), f_3(x)\}$ is a basis for P_2.

Proof. First, lets prove that $f_1(x), f_2(x), f_3(x)$ are linearly independent. Assume:

$$a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x) = 0, \quad x \in \mathbb{R}$$

Evaluate the above polynomial at $x = a$:

$$a_1 f_1(a) + a_2 f_2(a) + a_3 f_3(a) = a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 1 = a_3 = 0, \quad x \in \mathbb{R}$$

Similarly, evaluating the polynomial at b and c, we get that $a_1 = a_2 = 0$. Thus, they are linearly independent. Given that $f_1(x), f_2(x), f_3(x)$ are three linearly independent vectors on a space of dimension three, they generate the whole space, so they are a basis. □

(c) Let $f \in P_2$ and write f as a linear combination of $f_1(x), f_2(x), f_3(x)$:

$$f(x) = a_1 f_1(x) + a_2 f_2(x) + a_3 f_3(x)$$

What are the values of a_1, a_2 and a_3? That is, what are the "coordinates" of f with respect to $\mathbb{F}$?

Proof. Note again that:

$$f(a) = a_1 f_1(a) + a_2 f_2(a) + a_3 f_3(a) = a_1 \cdot 0 + a_2 \cdot 0 + a_3 \cdot 1 = a_3$$

$$f(b) = a_1 f_1(b) + a_2 f_2(b) + a_3 f_3(b) = a_1 \cdot 0 + a_2 \cdot 1 + a_3 \cdot 0 = a_2$$

$$f(c) = a_1 f_1(c) + a_2 f_2(c) + a_3 f_3(c) = a_1 \cdot 1 + a_2 \cdot 0 + a_3 \cdot 0 = a_1$$

That is, $a_1 = f(c), a_2 = f(b), a_3 = f(a)$.

□

6. **(20 points)** Let U and W be subspaces of a vector space V. Define:

$$U + W = \{u + w \quad | \quad u \in U, w \in W\}$$

Show that:

$$\dim(U) + \dim(W) = \dim(U + W) + \dim(U \cap W)$$

(Hint: use the Basis Extension Theorem).

Proof. Let $\{v_1, \ldots, v_k\}$ be a basis for $U \cap W$. By the Basis Extension Theorem, extend this to a basis $\{v_1, \ldots, v_k, u_1, \ldots, u_m\}$ of U, and to a basis $\{v_1, \ldots, v_k, w_1, \ldots, w_n\}$ of W. Then, $\dim(U \cap W) = k$, $\dim(U) = k + m$ and $\dim(W) = k + n$.

Lets show that $\{v_1, \ldots, v_k, u_1, \ldots, u_m, w_1, \ldots, w_n\}$ is a basis for $U + W$. Clearly, $U = span\{v_1, \ldots, v_k, u_1, \ldots, u_m\}$ and $W = span\{v_1, \ldots, v_k, w_1, \ldots, w_n\}$. It is easy to show that:

$$\begin{aligned} U + W &= span\{v_1, \ldots, v_k, u_1, \ldots, u_m\} + span\{v_1, \ldots, v_k, w_1, \ldots, w_n\} && (1) \\ &= span\{v_1, \ldots, v_k, u_1, \ldots, u_m, w_1, \ldots, w_n\} && (2) \end{aligned}$$

Lets show $v_1, \ldots, v_k, u_1, \ldots, u_m, w_1, \ldots, w_n$ are linearly independent. Assume:

$$a_1v_1 + \ldots + a_kv_k + b_1u_1 + \ldots + b_mu_m + c_1w_1 + \ldots + c_nw_n = 0$$

Note that:

$$a_1v_1 + \ldots + a_kv_k + b_1u_1 + \ldots + b_mu_m \in U$$

$$c_1w_1 + \ldots + c_nw_n \in W$$

Since:

$$a_1v_1 + \ldots + a_kv_k + b_1u_1 + \ldots + b_mu_m = -(c_1w_1 + \ldots + c_nw_n)$$

So,

$$a_1v_1 + \ldots + a_kv_k + b_1u_1 + \ldots + b_mu_m \in U \cap W$$

$$c_1w_1 + \ldots + c_nw_n \in U \cap W$$

By the enlargements of the basis $\{v_1, \ldots, v_k\}$, we know that $w_i \notin span\{v_1, \ldots, v_k\}$, so:

$$c_1w_1 + \ldots + c_nw_n \in U \cap W \iff c_i = 0, \quad \forall i \in \{1, \ldots, n\}$$

Given that $U \cap W$ is a subspace:

$$a_1v_1 + \ldots + a_kv_k + b_1u_1 + \ldots + b_mu_m \in U \cap W \iff b_1u_1 + \ldots + b_mu_m \in U \cap W$$

By the same argument as above, $b_i = 0$ for all $i \in \{1, \ldots, m\}$. Since $v_1, \ldots, v_k$ are linearly independent, $a_i = 0$. This completes the proof. □