ECON 897 Test (Week 3)
July 31, 2015

Important: This is a closed-book test. No books or lecture notes are permitted. You have **120** minutes to complete the test. Answer all questions. You can use all the results covered in class, but please make sure the conditions are satisfied. Write your name on each blue book and label each question clearly. Write legibly. Good luck!

1. (20 points) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice differentiable. Suppose there exists $\epsilon > 0$ such that $f''(x) > \epsilon$ for all $x \in \mathbb{R}$. Show that $f'(x) = 0$ for some $x \in \mathbb{R}$.

   Proof. Assume $f'(x_0) > 0$ for some $x_0 \in \mathbb{R}$. Since f is twice differentiable $\Rightarrow f'$ is differentiable so, by the Mean Value Theorem,

$$f'(x_0 + h) = f'(x_0) + f''(\zeta)h, \quad \zeta \in (x_0, x_0 + h)$$

   Note that for $h < 0$, $f''(\zeta) > \epsilon$ implies that $f''(\zeta)h < \epsilon h$. Thus,

$$f'(x_0 + h) = f'(x_0) + f''(\zeta)h < f'(x_0) + \epsilon h, \quad h < 0$$

   Taking h an arbitrarily large negative number, ($h < -\frac{f'(x_0)}{\epsilon} < 0$), the above inequality implies that $f'(x_0 + h) < 0$.

   So, take $h_0 < -\frac{f'(x_0)}{\epsilon}$. We have then that $f'(x_0) > 0$ and $f'(x_0 + h_0) < 0$. By the Intermediate Value Theorem, there must exist an $x_1 \in (x_0 + h_0, x_0)$ such that $f'(x_1) = 0$.

   □

2. (20 points) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be twice continuously differentiable. Assume there is a $c \in (a, b)$ such that $f'(c) = 0$ and $f''(c) < 0$. Show that f has a local maximum at c.

   Proof. Let $c \in (a, b)$ such that $f'(c) = 0$. Since f is twice differentiable, lets do a Taylor expansion of second order around the point c:

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(\zeta)}{2}(x - c)^2, \quad \zeta \in (x, c)$$

   Since f'' is continuous, there must exist a neighborhood $(c - \delta, c + \delta)$ such that $f''(x) < 0$, for all $x \in (c - \delta, c + \delta)$. Therefore, for all $x \in (c - \delta, c + \delta)$:

$$f(x) = f(c) + \underbrace{f'(c)}_{=0}(x-c) + \frac{\overbrace{f''(\zeta)}^{<0}}{2}(x-c)^2, \quad \zeta \in (x,c)$$

$$\Leftrightarrow \quad f(x) < f(c), \quad x \in (c-\delta, c+\delta)$$

Therefore, c is a local maximum.

$\square$

3. (20 points) Suppose that $f : (a,b) \longrightarrow \mathbb{R}$ is differentiable and f' is bounded. If $\{x_n\}$ is a sequence on (a,b) and $x_n \longrightarrow a$, then $f(x_n)$ converges.

Proof. <u>Note</u> that f is not defined at the point a!

f' is bounded, which means that $\sup_{x\in(a,b)} |f'(x)| < M$, for some $M \in \mathbb{R}$. By the Mean Value Theorem, for $x, y \in (a,b)$,

$$f(x) - f(y) = f'(\zeta)(x-y), \quad \zeta \in (x,y)$$

In particular,

$$|f(x_n) - f(x_m)| = |f'(\zeta)||x_n - x_m| \leq M|x_n - x_m|, \quad \zeta \in (x_n, x_m), \quad n, m \in \mathbb{Z}$$

Since, $\{x_n\}$ is a convergent sequence in $\mathbb{R}$, it is a Cauchy sequence, so for every $\epsilon > 0$ we can always find $N \in \mathbb{Z}$ such that $\forall n, m \geq N$, $|x_n - x_m| < \epsilon$. Thus, let $\epsilon/M > 0$. There exists $N \in \mathbb{Z}$ such that:

$$|f(x_n) - f(x_m)| \leq M|x_n - x_m| < M \cdot \frac{\epsilon}{M} = \epsilon, \quad \forall n, m \geq N$$

This means that $\{f(x_n)\}$ is a Cauchy sequence in $\mathbb{R}$, which is a complete metric space. Thus, $\{f(x_n)\}$ must converge.

$\square$

4. (20 points) State whether the following are linear subspaces, and prove your answer:

   (a) Let $W_n = \{f(x) \in P(F) | f(x) = 0$ or f(x) has degree <u>exactly equal</u> to $n > 1\}$. Is W_n a subspace of $P(F)$, where $P(F)$ is the space of polynomials?

Proof. It is not a subspace:

Take two polynomials of degree n with the same leading coefficient and subtract them. The resulting polynomial has degree $n-1$, so W_n is not closed under addition. For example, take:

$$x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 \in W_n$$
$$x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0 \in W_n$$

Subtracting them, it is clear that

$$(a_{n-1} - b_{n-1})x^{n-1} + \ldots + (a_1 - b_1)x + (a_0 - b_0) \notin W_n$$

□

(b) Let $A = \{(a_1, a_2, a_3) \in \mathbb{R}^3 | a_1 = a_3 + 2\}$. Is A a subspace of $\mathbb{R}^3$?

Proof. It is not a subspace:

It is not closed under addition. Take $(3, 0, 1) \in A$ and $(5, 0, 3) \in A$. Clearly, $(3, 0, 1) + (5, 0, 3) = (8, 0, 4) \notin A$.

□

5. (20 points) Let $A \in M_{n\times n}$, such that A^{-1} exists. Prove that the columns of A form a basis for $\mathbb{R}^n$.

Proof. A has an inverse, A^{-1}, so $AA^{-1} = I$. Let:

$$A^{-1} = \begin{pmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \ldots & b_{nn} \end{pmatrix}$$

And write A as an $n-$tuple of its column vectors: $A = (a_1, \ldots, a_n)$.

Then,

$$A \begin{pmatrix} b_{1i} \\ b_{2i} \\ \vdots \\ b_{ni} \end{pmatrix} = b_{1i}a_1 + \ldots + b_{ni}a_n = e_i$$

Which means that $e_i \in span\{a_1, \ldots, a_n\}$, for all $i \in \{1, \ldots, n\}$. Therefore, the columns of A span $\mathbb{R}^n$ and, since they are n vectors, they are a basis for $\mathbb{R}^n$.

$\square$

For the following problem, you cannot use the results in the exercises:

6. (20 points) If a is an $n \times 1$ vector, and b is a $1 \times m$ vector, prove that ab is an $n \times m$ matrix of rank at most equal to one.

Proof. The proof is the same as that of exercise 4 of July 29.

$\square$