Perturbation Methods V: Pruning

(Lectures on Solution Methods for Economists IX)

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October 24, 2018

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Introduction

Perturbation

- Advantages:
  1. Intuitive.
  2. Straightforward to compute.
  3. Fast.
  4. Accurate.
- Problem with simulations.

First-order approximation

- First-order approximation of a canonical RBC model without persistence in productivity shocks:

$$\widehat{k}_{t+1} = a_1 \widehat{k}_t + a_2 \varepsilon_t, \; \varepsilon_t \sim \mathcal{N}(0,1)$$

- Then:

$$\begin{aligned} \widehat{k}_{t+1} &= a_1 \left(a_1 \widehat{k}_{t-1} + a_2 \varepsilon_{t-1} \right) + a_2 \varepsilon_t \\ &= a_1^2 \widehat{k}_{t-1} + a_1 a_2 \varepsilon_{t-1} + a_2 \varepsilon_t \end{aligned}$$

- Since $a_1 < 1$ and assuming $\widehat{k}_0 = 0$

$$\widehat{k}_{t+1} = a_2 \sum_{j=0}^{t} a_1^j \varepsilon_{t-j}$$

which is a well-understood system.

Higher-order approximations

- Second-order approximation:

$$\widehat{k}_{t+1} = a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t, \ \varepsilon_t \sim \mathcal{N}(0,1)$$

- Then:

$$\begin{aligned}
\widehat{k}_{t+1} &= a_0 + a_1 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right) + a_2 \varepsilon_t \\
&\quad + a_3 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right)^2 + a_4 \varepsilon_t^2 \\
&\quad + a_5 \left(a_0 + a_1 \widehat{k}_t + a_2 \varepsilon_t + a_3 \widehat{k}_t^2 + a_4 \varepsilon_t^2 + a_5 \widehat{k}_t \varepsilon_t \right) \varepsilon_t
\end{aligned}$$

- We have terms in $\widehat{k}_t^3$ and $\widehat{k}_t^4$.

Problem

- For a large realization of ε_t, the terms in $\widehat{k}_t^3$ and $\widehat{k}_t^4$ make the system explode.
- This will happen as soon as we have a large simulation$\Rightarrow$ no unconditional moments would exist based on this approximation.
- This is true even when the corresponding linear approximation is stable.
- Then:
  1. How do you calibtate? (translation, spread, and deformation).
  2. How do you GMM or SMM?
  3. Asymptotics?

Solution

- For second-order approximations, Kim *et al.* (2008): pruning.

- Idea:

$$\begin{aligned}
\widehat{k}_{t+1} &= a_0 + a_1\left(a_0 + a_1\widehat{k}_t + a_2\varepsilon_t + a_3\widehat{k}_t^2 + a_4\varepsilon_t^2 + a_5\widehat{k}_t\varepsilon_t\right) + a_2\varepsilon_t \\
&\quad + a_3\left(a_0 + a_1\widehat{k}_t + a_2\varepsilon_t + a_3\widehat{k}_t^2 + a_4\varepsilon_t^2 + a_5\widehat{k}_t\varepsilon_t\right)^2 + a_4\varepsilon_t^2 \\
&\quad + a_5\left(a_0 + a_1\widehat{k}_t + a_2\varepsilon_t + a_3\widehat{k}_t^2 + a_4\varepsilon_t^2 + a_5\widehat{k}_t\varepsilon_t\right)\varepsilon_t
\end{aligned}$$

- We omit terms raised to powers higher than 2.

- Pruned approximation does not explode.

What do we do?

- Build a pruned state-space system.
- Apply pruning to an approximation of any arbitrary order.
- Prove that first and second unconditional moments exist.
- Closed-form expressions for first and second unconditional moments and IRFs.
- Conditions for the existence of some higher unconditional moments, such as skewness and kurtosis.
- Apply to a New Keynesian model with EZ preferences.
- Software available for distribution.

Consequences

1. GMM and IRF-matching can be implemented without simulation.

2. First and second unconditional moments or IRFs can be computed in a trivial amount of time for medium-sized DSGE models approximated up to third-order.

3. Use the unconditional moment conditions in optimal GMM estimation to build a limited information likelihood function for Bayesian inference (Kim, 2002).

4. Foundation for indirect inference as in Smith (1993) and SMM as in Duffie and Singleton (1993).

5. Calibration.

State-Space Representations

Dynamic models and state-space representations

- Dynamic model:

$$\mathbf{x}_{t+1} = \mathbf{h}\left(\mathbf{x}_t, \sigma\right) + \sigma \eta \epsilon_{t+1},\ \epsilon_{t+1} \sim IID\left(\mathbf{0}, \mathbf{I}\right)$$
$$\mathbf{y}_t = \mathbf{g}\left(\mathbf{x}_t, \sigma\right)$$

- What is behind this system?

- General structure (use of augmented state vector).

The state-space system I

- Perturbation methods approximate $\mathbf{h}(\mathbf{x}_t, \sigma)$ and $\mathbf{g}(\mathbf{x}_t, \sigma)$ with Taylor-series expansions around $\mathbf{x}_{ss} = \sigma = 0$.

- A first-order approximated state-space system replaces $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ with $\mathbf{g_x x}_t$ and $\mathbf{h_x x}_t$.

- If $\forall$ $\mathbf{mod}(eig(\mathbf{h_x})) < 1$, the approximation fluctuates around the steady state (also its mean value).

- Thus, easy to calibrate the model based on first and second moments or to estimate it using Bayesian methods, MLE, GMM, SMM, etc.

The state-space system II

- We can replace $\mathbf{g}(\mathbf{x}_t, \sigma)$ and $\mathbf{h}(\mathbf{x}_t, \sigma)$ with their higher-order Taylor-series expansions.

- However, the approximated state-space system cannot, in general, be shown to have any finite moments.

- Also, it often displays explosive dynamics.

- This occurs even with simple versions of the New Keynesian model.

- Hence, it is difficult to use the approximated state-space system to calibrate or to estimate the parameters of the model.

The pruning method: second-order approximation I

- Partition states:

$$\begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^s\right)' \end{bmatrix}$$

- Original state-space representation:

$$\mathbf{x}_{t+1}^{(2)} = \mathbf{h}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\left(\mathbf{x}_t^f + \mathbf{x}_t^s\right) \otimes \left(\mathbf{x}_t^f + \mathbf{x}_t^s\right)\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \sigma\eta\epsilon_{t+1}$$

$$\mathbf{y}_t^{(2)} = \mathbf{g}_{\mathbf{x}}\mathbf{x}_t^{(2)} + \frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t^{(2)} \otimes \mathbf{x}_t^{(2)}\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

The pruning method: second-order approximation II

- New state-space representation:

$$\mathbf{x}^f_{t+1} = \mathbf{h_x}\mathbf{x}^f_t + \sigma\eta\epsilon_{t+1}$$
$$\mathbf{x}^s_{t+1} = \mathbf{h_x}\mathbf{x}^s_t + \frac{1}{2}\mathbf{H_{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2$$
$$\mathbf{y}^f_t = \mathbf{g_x}\mathbf{x}^f_t$$
$$\mathbf{y}^s_t = \mathbf{g_x}\left(\mathbf{x}^f_t + \mathbf{x}^s_t\right) + \frac{1}{2}\mathbf{G_{xx}}\left(\mathbf{x}^f_t \otimes \mathbf{x}^f_t\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2$$

- All variables are second-order polynomials of the innovations.

The pruning method: third-order approximation I

- Partition states:

$$\left[\begin{array}{ccc} \left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^{rd}\right)' \end{array}\right]$$

- Original state-space representation:

$$\begin{aligned}
\mathbf{x}_{t+1}^{(3)} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^{(3)} + \frac{1}{2}\mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \frac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\
&\quad + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^{(3)} + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 + \sigma\eta\epsilon_{t+1} \\
\mathbf{y}_t^{(3)} &= \mathbf{g}_{\mathbf{x}}\mathbf{x}_t^{(3)} + \frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) + \frac{1}{6}\mathbf{G}_{\mathbf{xxx}}\left(\mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)} \otimes \mathbf{x}_t^{(3)}\right) \\
&\quad + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^{(3)} + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3
\end{aligned}$$

The pruning method: third-order approximation II

- New state-space representation:

$$
\begin{aligned}
& \text{Second-order pruned state-space representation}+ \\
\mathbf{x}_{t+1}^{rd} &= \mathbf{h}_{\mathbf{x}}\mathbf{x}_t^{rd} + \mathbf{H}_{\mathbf{xx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right) + \frac{1}{6}\mathbf{H}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}^f\right) \\
&\quad + \frac{3}{6}\mathbf{h}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{h}_{\sigma\sigma\sigma}\sigma^3 \\
\mathbf{y}_t^{rd} &= \mathbf{g}_{\mathbf{x}}\left(\mathbf{x}_t^f + \mathbf{x}_t^s + \mathbf{x}_t^{rd}\right) + \frac{1}{2}\mathbf{G}_{\mathbf{xx}}\left(\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + 2\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^s\right)\right) \\
&\quad + \frac{1}{6}\mathbf{G}_{\mathbf{xxx}}\left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f \otimes \mathbf{x}_t^f\right) + \frac{1}{2}\mathbf{g}_{\sigma\sigma}\sigma^2 + \frac{3}{6}\mathbf{g}_{\sigma\sigma\mathbf{x}}\sigma^2\mathbf{x}_t^f + \frac{1}{6}\mathbf{g}_{\sigma\sigma\sigma}\sigma^3
\end{aligned}
$$

- All variables are third-order polynomials of the innovations.

Higher-order approximations

- We can generalize previous steps:

1. Decompose the state variables into first-, second-, ... , and kth-order effects.
2. Set up laws of motions for the state variables capturing only first-, second-, ... , and kth-order effects.
3. Construct the expression for control variables by preserving only effects up to kth-order.

Statistical properties: second-order approximation I

Theorem
If $\forall\ \mathbf{mod}\left(eig\left(\mathbf{h_x}\right)\right) < 1$ *and* ϵ_{t+1} *has finite fourth moments, the pruned state-space system has finite first and second moments.*

Theorem
If $\forall\ \mathbf{mod}\left(eig\left(\mathbf{h_x}\right)\right) < 1$ *and* ϵ_{t+1} *has finite sixth and eighth moments, the pruned state-space system has finite third and fourth moments.*

Statistical properties: second-order approximation I

- We introduce the vectors

$$\mathbf{z}_t^{(2)} \equiv \begin{bmatrix} \left(\mathbf{x}_t^f\right)' & \left(\mathbf{x}_t^s\right)' & \left(\mathbf{x}_t^f \otimes \mathbf{x}_t^f\right)' \end{bmatrix}'$$

$$\xi_{t+1}^{(2)} \equiv \begin{bmatrix} \epsilon_{t+1} \\ \epsilon_{t+1} \otimes \epsilon_{t+1} - vec\left(\mathbf{I}_{n_e}\right) \\ \epsilon_{t+1} \otimes \mathbf{x}_t^f \\ \mathbf{x}_t^f \otimes \epsilon_{t+1} \end{bmatrix}$$

- First moment:

$$\mathbb{E}\left[\mathbf{x}_t^{(2)}\right] = \underbrace{\mathbb{E}\left[\mathbf{x}_t^f\right]}_{=0} + \underbrace{\mathbb{E}\left[\mathbf{x}_t^s\right]}_{\neq 0}$$

$$\mathbb{E}\left[\mathbf{x}_t^s\right] = \left(\mathbf{I} - \mathbf{h_x}\right)^{-1}\left(\frac{1}{2}\mathbf{H_{xx}}\left(\mathbf{I} - \mathbf{h_x} \otimes \mathbf{h_x}\right)^{-1}\left(\sigma\eta \otimes \sigma\eta\right) vec\left(\mathbf{I}_{n_e}\right) + \frac{1}{2}\mathbf{h}_{\sigma\sigma}\sigma^2\right)$$

$$\mathbb{E}\left[\mathbf{y}_t^s\right] = \mathbf{C}^{(2)}\mathbb{E}\left[\mathbf{z}_t^{(2)}\right] + \mathbf{d}^{(2)}$$

Statistical properties: second-order approximation III

- Second moment:

$$\mathbb{V}\left(\mathbf{z}_t^{(2)}\right) = \mathbf{A}^{(2)}\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)\left(\mathbf{A}^{(2)}\right)' + \mathbf{B}^{(2)}\mathbb{V}\left(\xi_t^{(2)}\right)\left(\mathbf{B}^{(2)}\right)'$$

$$Cov\left(\mathbf{z}_{t+l}^{(2)}, \mathbf{z}_t^{(2)}\right) = \left(\mathbf{A}^{(2)}\right)^l \mathbb{V}\left(\mathbf{z}_t^{(2)}\right) \quad \text{for } l = 1, 2, 3, \ldots$$

$$\mathbb{V}\left[\mathbf{x}_t^{(2)}\right] = \mathbb{V}\left(\mathbf{x}_t^f\right) + \mathbb{V}\left(\mathbf{x}_t^s\right) + Cov\left(\mathbf{x}_t^f, \mathbf{x}_t^s\right) + Cov\left(\mathbf{x}_t^s, \mathbf{x}_t^f\right)$$

$$\mathbb{V}\left[\mathbf{y}_t^s\right] = \mathbf{C}^{(2)}\mathbb{V}\left[\mathbf{z}_t\right]\left(\mathbf{C}^{(2)}\right)'$$

$$Cov\left(\mathbf{y}_t^s, \mathbf{y}_{t+l}^s\right) = \mathbf{C}^{(2)} Cov\left(\mathbf{z}_{t+l}^{(2)}, \mathbf{z}_t^{(2)}\right)\left(\mathbf{C}^{(2)}\right)' \quad \text{for } l = 1, 2, 3, \ldots$$

where we solve for $\mathbb{V}\left(\mathbf{z}_t^{(2)}\right)$ by standard methods for discrete Lyapunov equations.

Statistical properties: second-order approximation IV

- Generalized impulse response function (GIRF): Koop *et al.* (1996)

$$GIRF_{\mathbf{var}}\left(l,\nu,\mathbf{w}_t\right)=\mathbb{E}\left[\mathbf{var}_{t+l}|\mathbf{w}_t,\epsilon_{t+1}=\nu\right]-\mathbb{E}\left[\mathbf{var}_{t+l}|\mathbf{w}_t\right]$$

- Importance in models with volatility shocks.

Statistical properties: third-order approximation I

Theorem
If $\forall$ $\mathbf{mod}\left(eig\left(\mathbf{h_x}\right)\right) < 1$ *and* ϵ_{t+1} *has finite sixth moments, the pruned state-space system has finite first and second moments.*

Theorem
If $\forall$ $\mathbf{mod}\left(eig\left(\mathbf{h_x}\right)\right) < 1$ *and* ϵ_{t+1} *has finite ninth and twelfth moments, the pruned state-space system has finite third and fourth moments.*

- Similar (but long!!!!!) formulae for first and second moments and IRFs.

Application

Application I

- A middle-scale New Keynesian model with habit formation and EZ preferences.
- Why?
    1. Standard model for policy analysis.
    2. Sizable higher-order terms.
    3. The model should not generate explosive sample paths when simulated with the unpruned state-space system.

Application II

- What will we do?

    1. Check the accuracy of pruned state-space representations.
    2. Estimate the model with GMM and SMM.
    3. Explore its properties.

Households I

- Preferences:

$$V_t \equiv \begin{cases} u_t + \beta \left(\mathbb{E}_t \left[V_{t+1}^{1-\phi_3} \right] \right)^{\frac{1}{1-\phi_3}} & \text{if } u_t > 0 \text{ for all } t \\ u_t - \beta \left(\mathbb{E}_t \left[\left(-V_{t+1} \right)^{1-\phi_3} \right] \right)^{\frac{1}{1-\phi_3}} & \text{if } u_t < 0 \text{ for all } t \end{cases}$$

where

$$u_t \equiv d_t \frac{\left(c_t - bc_{t-1} \right)^{1-\phi_2}}{1-\phi_2} + \left(z_t^* \right)^{(1-\phi_2)} \phi_0 \frac{\left(1 - h_t \right)^{1-\phi_1}}{1-\phi_1}$$

and

$$\log d_{t+1} = \rho_d \log d_t + \epsilon_{d,t+1}, \;\; \epsilon_{d,t} \sim \mathcal{IID}\left(0, \sigma_d^2\right)$$

Households II

- The budget constraint:

$$c_t + \frac{i_t}{\Upsilon_t} + \int D_{t,t+1} x_{t+1} d\omega_{t,t+1} = w_t h_t + r_t^k k_t + \frac{x_t}{\pi_t} + div_t$$

- Capital:

$$k_{t+1} = (1-\delta)\, k_t + i_t - \frac{\kappa}{2} \left(\frac{i_i}{k_t} - \psi \right)^2 k_t$$

Firms

- Final firm:

$$y_t = \left(\int_0^1 y_{i,t}^{(\eta-1)/\eta} di \right)^{\eta/(\eta-1)}$$

- Intermediate goods producers:

$$y_{i,t} = a_t k_{i,t}^{\theta} \left(z_t h_{i,t} \right)^{1-\theta}$$

$$\log z_{t+1} = \log z_t + \log \mu_{z,ss}$$

$$z_t^* \equiv \Upsilon_t^{\frac{\theta}{1-\theta}} z_t$$

$$\log a_{t+1} = \rho_a \log a_t + \epsilon_{a,t+1}, \ \epsilon_{a,t} \sim \mathcal{IID}\left(0, \sigma_a^2\right)$$

- Two versions: Calvo and quadratic price adjustment costs $\xi_p \geq 0$ w.r.t. π_{ss}.

Government

- Taylor rule for the monetary authority:

$$r_{t,1} = (1 - \rho_r)\, r_{ss} + \rho_r\, r_{t-1,1} + \beta_\pi \log\left(\frac{\pi_t}{\pi_{ss}}\right) + \beta_y \log\left(\frac{y_t}{z_t^* Y_{ss}}\right)$$

Solution

- 13 state variables.
- We detrend variables.
- Second- and third-order approximation.
- We check the Euler equation errors to compare the accuracy of pruned and unpruned state-space systems with a standard calibration.
- Results are even better for large innovations.

	First-order	Second-order	
	RMSE	RMSE[P]	RMSE
Benchmark			
Household's value function	1.0767	0.9382	**0.5998**
Household's FOC for consumption	49.8994	1.8884	**1.8069**
Household's FOC for capital	5.5079	0.2006	**0.1976**
Household's FOC for labor	0.4605	**0.1769**	0.5305
Household's FOC for investment	0.0570	**0.0092**	0.1023
Euler-eq. for one-period interest rate	4.9705	0.1930	**0.1922**
Firm's FOC for prices	3.8978	0.2026	**0.1709**
Income identity	0.1028	**0.1004**	0.1840
Law of motion for capital	0.1295	**0.0239**	0.2499
Average error	7.3447	**0.4148**	0.4482

	First-order	Third-order	
	RMSE	RMSE^{P}	RMSE
Benchmark			
Household's value function	1.0767	0.4801	**0.3269**
Household's FOC for consumption	49.8994	**0.7840**	1.1971
Household's FOC for capital	5.5079	**0.0907**	0.1473
Household's FOC for labor	0.4605	**0.0406**	0.1831
Household's FOC for investment	0.0570	**0.0023**	0.0553
Euler-eq. for one-period interest rate	4.9705	**0.0770**	0.1320
Firm's FOC for prices	3.8978	**0.0903**	0.0845
Income identity	0.1028	**0.0405**	0.1333
Law of motion for capital	0.1295	**0.0056**	0.1327
Average error	7.3447	**0.1790**	0.2658

Estimation

- Version with Calvo pricing.
- US Macro and financial data from 1961Q3 to 2007Q4:
    1. consumption growth Δc_t
    2. investment growth Δi_t
    3. inflation π_t
    4. 1-quarter nominal interest rate $r_{t,1}$
    5. 10-year nominal interest rate $r_{t,40}$
    6. 10-year ex post excess holding period return $xhr_{t,40} \equiv \log\left(P_{t,39}/P_{t-1,40}\right) - r_{t-1,1}$
    7. log of hours $\log h_t$.
- Use GMM and SMM.
- Computation: 0.03 seconds in second-order, 0.8 seconds in third-order, 1.4 for GIRFs (all in `Matlab`).

	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
β	0.9925 (0.0021)	0.9926 (0.0002)	0.9926 (0.0023)
b	0.6889 (0.0194)	0.7137 (0.0004)	0.7332 (0.0085)
h_{ss}	0.3402 (0.0010)	0.3401 (0.0004)	0.3409 (0.0065)
ϕ_1	6.1405 (1.2583)	6.1252 (0.0002)	6.1169 (0.0040)
ϕ_2	1.5730 (0.1400)	1.5339 (0.0008)	1.5940 (0.0009)
ϕ_3	-196.31 (51.90)	-197.36 (0.01)	-194.22 (0.01)
κ	4.1088 (0.7213)	3.5910 (0.0160)	3.5629 (0.1085)
α	0.9269 (0.0044)	0.9189 (0.0026)	0.9195 (0.0024)
ρ_r	0.6769 (0.6086)	0.6759 (0.0723)	0.6635 (0.1464)
β_π	3.9856 (8.2779)	3.6974 (0.7892)	3.6216 (1.8555)
β_y	0.5553 (1.5452)	0.50691 (0.1465)	0.5027 (0.3685)

	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
$\mu_{\Upsilon,ss}$	1.0018 (0.0012)	1.0017 (0.0007)	1.0016 (0.0006)
$\mu_{z,ss}$	1.0050 (0.0005)	1.0051 (0.0004)	1.0052 (0.0003)
ρ_a	0.9192 (0.0081)	0.9165 (0.0030)	0.9139 (0.0036)
ρ_d	0.9915 (0.0023)	0.9914 (0.0005)	0.9911 (0.0019)
π_{ss}	1.0407 (0.0134)	1.0419 (0.0022)	1.0432 (0.0057)
σ_α	0.0171 (0.0006)	0.0183 (0.0005)	0.0183 (0.0003)
σ_d	0.0144 (0.0017)	0.0144 (0.0005)	0.0143 (0.0018)
skew_a	—	—	0.2296 (0.0298)
tail_a	—	—	1.2526 (0.0437)
skew_d	—	—	0.0693 (0.4530)
tail_d	—	—	1.1329 (3.4724)

	Data	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
Means				
$\Delta c_t \times 100$	2.439	2.399	2.429	2.435
$\Delta i_t \times 100$	3.105	3.111	3.099	3.088
$\pi_t \times 100$	3.757	3.681	3.724	3.738
$r_{t,1} \times 100$	5.605	5.565	5.548	5.582
$r_{t,40} \times 100$	6.993	6.925	6.955	6.977
$xhr_{t,40} \times 100$	1.724	1.689	1.730	1.717
$\log h_t$	-1.084	-1.083	-1.083	-1.083

	Data	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
Standard deviations (in pct)				
Δc_t	2.685	1.362	1.191	1.127
Δi_t	8.914	8.888	8.878	8.944
π_t	2.481	3.744	3.918	3.897
$r_{t,1}$	2.701	4.020	4.061	4.060
$r_{t,40}$	2.401	2.325	2.326	2.308
$xhr_{t,40}$	22.978	22.646	22.883	22.949
$\log h_t$	1.676	3.659	3.740	3.721

	Data	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
Auto-correlations: 1 lag				
$corr\,(\Delta c_t, \Delta c_{t-1})$	0.254	0.702	0.726	0.7407
$corr\,(\Delta i_t, \Delta i_{t-1})$	0.506	0.493	0.480	0.4817
$corr\,(\pi_t, \pi_{t-1})$	0.859	0.988	0.986	0.9861
$corr\,(r_{t,1}, r_{t-1,1})$	0.942	0.989	0.987	0.987
$corr\,(r_{t,40}, r_{t-1,40})$	0.963	0.969	0.969	0.968
$corr\,(xhr_{t,40}, xhr_{t-1,40})$	-0.024	0.000	-0.003	-0.003
$corr\,(\log h_t, \log h_{t-1})$	0.792	0.726	0.678	0.6706

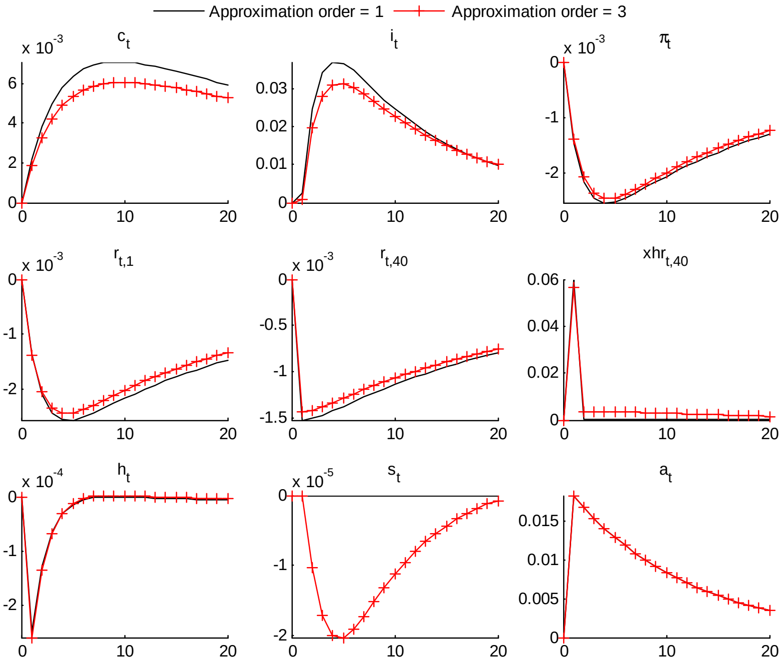
	Data	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
Skewness				
Δc_t	-0.679	0.024	0.034	0.193
Δi_t	-0.762	-0.191	-0.254	-0.122
π_t	1.213	0.013	0.014	-0.054
r_t	1.053	0.012	0.011	-0.051
$r_{t,40}$	0.967	0.014	0.017	-0.043
$xhr_{t,40}$	0.364	-0.026	-0.028	0.368

	Data	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
Kurtosis				
Δc_t	5.766	3.011	3.015	3.547
Δi_t	5.223	3.157	3.279	4.425
π_t	4.232	2.987	2.985	3.040
r_t	4.594	2.968	2.975	3.033
$r_{t,40}$	3.602	2.987	2.979	3.028
$xhr_{t,40}$	5.121	3.003	3.006	5.167

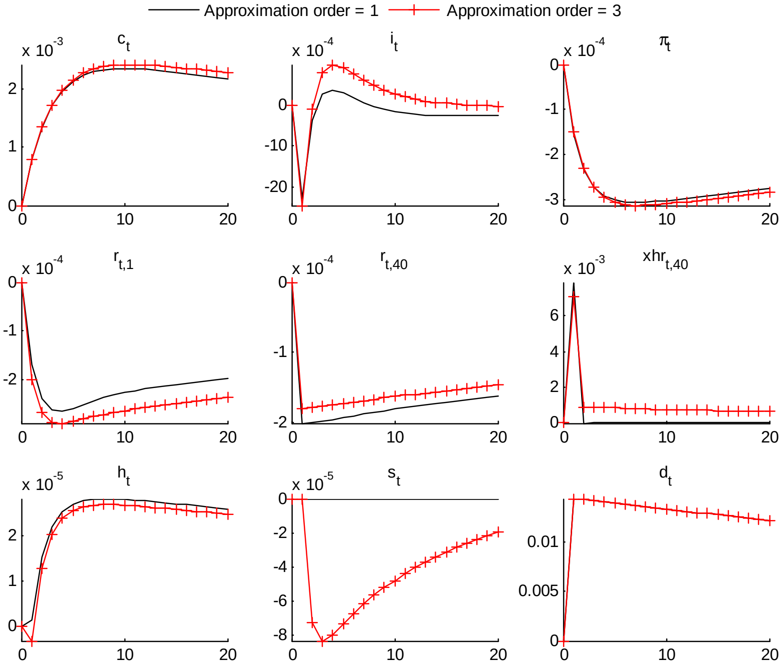
	Data	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
$corr\left(\Delta c_t, \Delta i_t\right)$	0.594	0.590	0.579	0.582
$corr\left(\Delta c_t, \pi_t\right)$	-0.362	-0.238	-0.296	-0.310
$corr\left(\Delta c_t, r_{t,1}\right)$	-0.278	-0.210	-0.274	-0.290
$corr\left(\Delta c_t, r_{t,40}\right)$	-0.178	-0.3337	-0.355	-0.366
$corr\left(\Delta c_t, xhr_{t,40}\right)$	0.271	0.691	0.655	0.641
$corr\left(\Delta c_t, \log h_t\right)$	0.065	-0.677	-0.670	-0.674
$corr\left(\Delta i_t, \pi_t\right)$	-0.242	-0.075	-0.098	-0.098
$corr\left(\Delta i_t, r_{t,1}\right)$	-0.265	-0.058	-0.084	-0.088
$corr\left(\Delta i_t, r_{t,40}\right)$	-0.153	-0.130	-0.133	-0.135
$corr\left(\Delta i_t, xhr_{t,40}\right)$	0.021	0.015	0.024	0.027
$corr\left(\Delta i_t, \log h_t\right)$	0.232	-0.398	-0.406	-0.418

	Data	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
$corr\left(\pi_t, r_{t,1}\right)$	0.628	0.994	0.997	0.997
$corr\left(\pi_t, r_{t,40}\right)$	0.479	0.990	0.988	0.987
$corr\left(\pi_t, xhr_{t,40}\right)$	-0.249	-0.130	-0.142	-0.141
$corr\left(\pi_t, \log h_t\right)$	-0.467	0.132	0.128	0.154
$corr\left(r_{t,1}, r_{t,40}\right)$	0.861	0.986	0.991	0.991
$corr\left(r_{t,1}, xhr_{t,40}\right)$	-0.233	-0.122	-0.137	-0.138
$corr\left(r_{t,1}, \log h_t\right)$	-0.369	0.177	0.153	0.180
$corr\left(r_{t,40}, xhr_{t,40}\right)$	-0.121	-0.247	-0.248	-0.249
$corr\left(r_{t,40}, \log h_t\right)$	-0.409	0.229	0.238	0.268
$corr\left(xhr_{t,40}, \log h_t\right)$	-0.132	-0.644	-0.680	-0.690

	GMM^{2nd}	GMM^{3rd}	SMM^{3rd}
Objective function: Q	0.0920	0.1055	0.0958
Number of moments	42	42	54
Number of parameters	18	18	22
P-value	0.8437	0.7183	0.9797



Approximation order = 1
Approximation order = 3
c_t
i_t
π_t
$r_{t,1}$
$r_{t,40}$
$xhr_{t,40}$
h_t
s_t
a_t
x 10^{-3}
x 10^{-4}
x 10^{-5}


Approximation order = 1
Approximation order = 3
c_t
x 10^{-3}
i_t
x 10^{-4}
π_t
x 10^{-4}
$r_{t,1}$
x 10^{-4}
$r_{t,40}$
x 10^{-4}
$xhr_{t,40}$
x 10^{-3}
h_t
x 10^{-5}
s_t
x 10^{-5}
d_t